

Chapter 4:

*Area,
Perimeter,
and Volume*



Introduction

The performance tasks in this chapter focus on applying the properties of triangles and polygons to compute area, perimeter, and volume.

Boxing Basketballs

1. A basketball (sphere) with a circumference of approximately 30 inches is packed in a box (cube) so that it touches each side of the interior of the box. Answer the following question, ignoring the thickness of the surface of the ball and the surface of the box.

What is the volume of the wasted space in the box?

2. A box (cube) that has a side length of 9.5 inches is packed inside a ball (sphere) so that the corners of the box touch the interior of the sphere. Answer the following question, ignoring the thickness of the surface of the ball and the surface of the box.

What is the volume of the wasted space in the ball?



Notes

Materials:

One pencil and graphing calculator per student

Geometry TEKS Focus:

(G.5) **Geometric patterns.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to:

(D) identify and apply patterns from right triangles to solve meaningful problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

(G.8) Congruence and the geometry of size.

The student uses tools to determine measurements of geometric figures and extends measurement concepts to find perimeter, area, and volume in problem situations.

The student is expected to:

(D) find surface areas and volumes of prisms, pyramids, spheres, cones, cylinders, and composites of these figures in problem situations.

Additional Geometry TEKS:

(G.8) **Congruence and the geometry of size.** The student uses tools to determine measurements of geometric figures and extends measurement concepts to find area, perimeter, and volume in problem situations.

The student is expected to:

(C) derive, extend, and use the Pythagorean Theorem; and

Teacher Notes

Scaffolding Questions:

- For the first situation, how do the cube and the sphere come in contact with each other (sides or vertices)?
- Do the cube and the sphere have any common measurements?
- In the second problem, how do the cube and the sphere come in contact with each other?
- What triangle is formed by the diagonal of the base of the cube and the edges of the base of the cube?
- What triangle is formed by the diagonal of the cube, the diagonal of the base, and one vertical edge of the cube?

Sample Solutions:

1. First, find the volumes of the ball and the box. The diameter of the ball is the side length of the box. Use the circumference, C , to find the diameter, d .

$$C = \pi d$$

$$30 = \pi d$$

$$d \approx 9.55 \text{ in}$$

The volume of the box, V , in terms of the length of the side, s , is found by using the formula $V = s^3$.

$$V = 9.55^3$$

$$V = 870.98 \text{ in}^3$$

Find the volume of the ball.

$$V = \frac{4}{3} \pi r^3$$

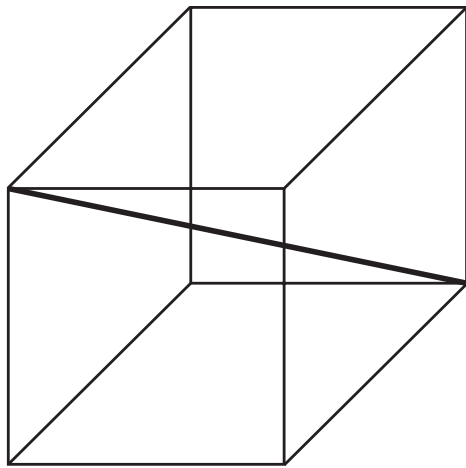
$$V = \frac{4}{3} \pi \left(\frac{9.55}{2} \right)^3$$

$$V \approx 456.05 \text{ in}^3$$

Subtract the volume of the ball from the volume of the box to find the volume of wasted space.

$$870.98 \text{ in}^3 - 456.05 \text{ in}^3 = 414.93 \text{ in}^3$$

2. If the cube is surrounded by the ball, the cube will touch the interior of the sphere at eight vertex points. To determine the diameter of the sphere we must find the diagonal of the cube. Since the diagonal of the cube passes through the center of the sphere and its endpoints touch the sphere, its length is the diameter of the sphere.



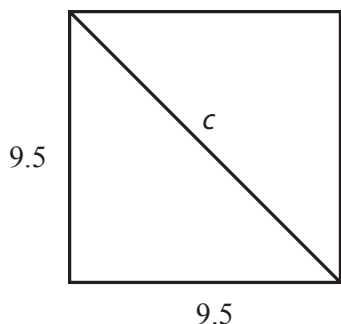
This can be found by using the Pythagorean Theorem or 45-45-90 special right triangles.

Connections to TAKS:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 8: The student will demonstrate an understanding of the concepts and uses of

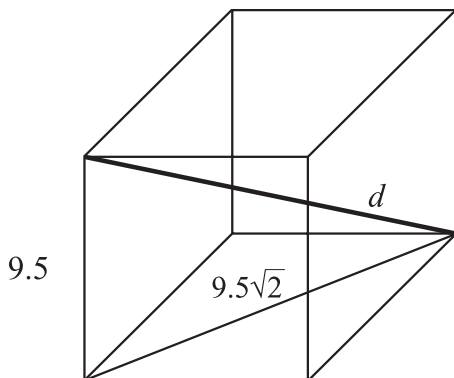
First find a diagonal of a side of the cube. The right triangle has sides measuring 9.5 inches.



$$9.5^2 + 9.5^2 = c^2$$

$$c = \sqrt{2(9.5^2)} = 9.5\sqrt{2}$$

Now use the Pythagorean Theorem again to find the diagonal of the cube. This diagonal is the hypotenuse of a right triangle. One leg of the triangle is the side of the square, which measures 9.5 inches, and the other leg is the diagonal of the side face, $9.5\sqrt{2}$ inches.



$$9.5^2 + (9.5\sqrt{2})^2 = d^2$$

$$d \approx 16.45 \text{ inches}$$

Now find the volume of the sphere and the cube.

Cube

$$V = s^3$$

$$V = 9.5^3$$

$$V = 857.375 \text{ in}^3 \approx 857.38 \text{ in}^3$$

Sphere

$$V = \frac{4}{3}\pi r^3$$

$$V = \frac{4}{3}\pi \left(\frac{16.45}{2}\right)^3$$

$$V \approx 2330.75 \text{ in}^3$$

The volume of the wasted space is the volume of the sphere minus the volume of the cube.

$$2330.75 - 857.38 = 1473.37$$

The volume of the wasted space is approximately 1473.37 in^3 .

Extension Questions:

- If the circumference of the ball were doubled in problem 1, how would the wasted space be affected?

If the circumference is doubled, the diameter and the radius would also be doubled. The volume should be multiplied by 2^3 or 8. This idea may be demonstrated by looking at the general formula.

The volume of the wasted space, in terms of the diameter for the original situation, is

$$d^3 - \frac{4}{3}\pi\left(\frac{d}{2}\right)^3 = d^3\left[1 - \frac{4}{3}\pi\left(\frac{1}{2}\right)^3\right]$$

If the diameter is doubled, the equation becomes

$$(2d)^3 - \frac{4}{3}\pi\left(\frac{2d}{2}\right)^3 = (2d)^3\left[1 - \frac{4}{3}\pi\left(\frac{1}{2}\right)^3\right] = 8d^3\left[1 - \frac{4}{3}\pi\left(\frac{1}{2}\right)^3\right]$$

The wasted space is multiplied by 8.

- If the length of the side of the box in problem 2 is multiplied by one-third, how will the diameter of the sphere be affected?

The diameter will also be multiplied by one-third. The diameter of the sphere is the diagonal of the cube. The diagonal of the cube is the hypotenuse of a right triangle. The right triangle is similar to the right triangle in the original cube. The corresponding sides of the two right triangles are proportional. Since the side of the new cube (also the leg of the right triangle) is one-third times the side of the original cube, the diagonal of the new cube (also the hypotenuse of the similar triangle) is one-third times the diagonal of the original cube.

Student Work Sample

The work on the next page was performed as an individual assessment following a unit on volumes. The student's work exemplifies many of the criteria from the solution guide. Note the following criteria:

- Shows an understanding of the relationships among elements.

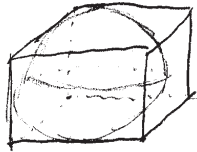
The student uses arrows in the first problem to show that circumference is used to determine the diameter, which is equal to the length of the box's edges. In the second problem, he shows how he used the edge of the box and the right triangles to find the radius of the sphere. He shows how he determines the answers in both problems by subtracting the two volumes, the volume of the box, and the volume of the sphere.

- States a clear and accurate solution using correct units.

The student not only uses the correct units—"in³"—but indicates that it is the volume of the wasted space.

Boxing Basketballs

1



Circumference of a Basketball = 30 in

So the diameter of it is = $\frac{30}{\pi}$ = the length of box's edges

the radius of it is = $\frac{30}{2\pi} = \frac{15}{\pi}$ in

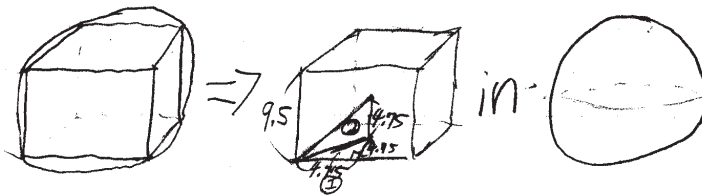
$$V \text{ of the box} = \left(\frac{30}{\pi}\right)^3$$

$$V \text{ of the ball} = \left(\frac{15}{\pi}\right)^3 \pi \times \frac{4}{3}$$

$$V \text{ of the wasted space} = \left(\frac{30}{\pi}\right)^3 - \left(\frac{15}{\pi}\right)^3 \pi \times \frac{4}{3}$$

$$= \boxed{444.85 \text{ in}^3}$$

2



To find the radius of the sphere (look at the diagram):

① Use special triangle: $4.75\sqrt{2}$

then ② use Pythagorean theorem: $\sqrt{4.75^2 + (4.75\sqrt{2})^2}$

This is the radius of the sphere

$$V \text{ of the box} = 9.5^3$$

$$V \text{ of the sphere} = \left(\sqrt{4.75^2 + (4.75\sqrt{2})^2}\right)^3 \times 3.14 \times \frac{4}{3}$$

$$V \text{ of the wasted space} = \left(\sqrt{4.75^2 + (4.75\sqrt{2})^2}\right)^3 \times 3.14 \times \frac{4}{3} - 9.5^3$$

$$= \boxed{1474.10 \text{ in}^3}$$

Flower

A landscape company has produced the garden design below (see figure 1). The petals are constructed by drawing circles from the vertex to the center of a regular hexagon with a radius of 20 feet (see figure 2).

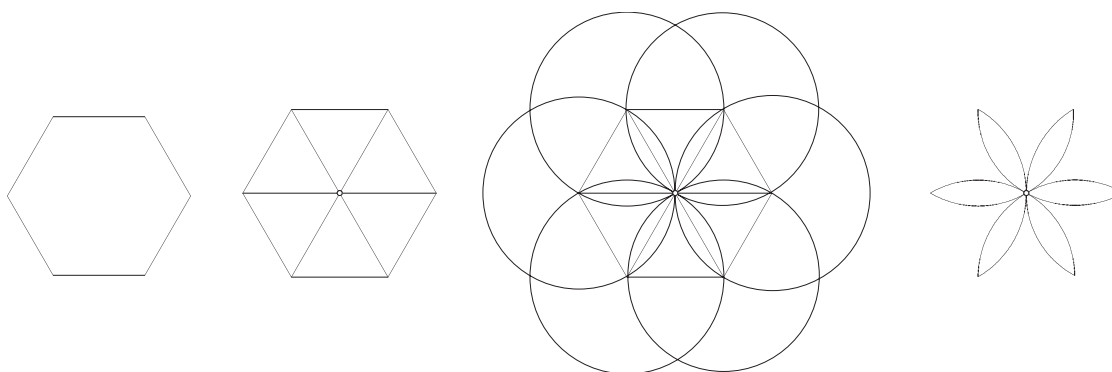
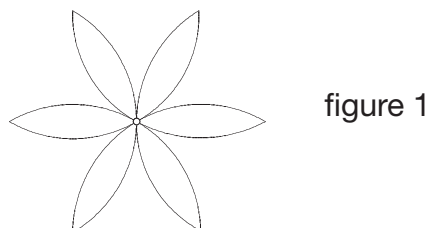


figure 2

The company plans to plant flowers inside the area of each “petal” of the garden design. The company must know the area of the petals in order to determine how many flowers to purchase for planting. Find the area of the petals. Give answers correct to the nearest hundredth.



Notes

measurement and similarity.

Materials:

One graphing calculator per student

Geometry TEKS Focus:**(G.8) Congruence and the geometry of size.**

The student uses tools to determine measurements of geometric figures and extends measurement concepts to find perimeter, area, and volume in problem situations.

The student is expected to:

(A) find areas of regular polygons, circles, and composite figures;

(B) find areas of sectors and arc lengths of circles using proportional reasoning;

Additional Geometry TEKS:

(G.5) **Geometric patterns.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to:

(D) identify and apply patterns from right triangles to solve meaningful problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

Teacher Notes

Scaffolding Questions:

- Are all of the petals the same size?
- What is the combination of figures that form a petal?
- How can the measure of the arc be determined?

Sample Solutions:

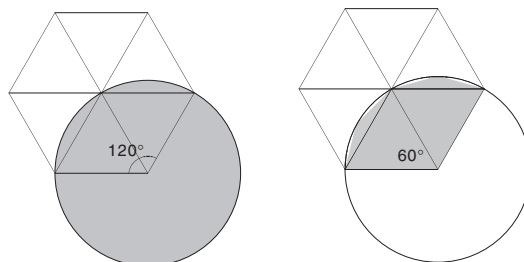
The hexagon eases the solution in that it can be subdivided into six equilateral triangles.

First, find the area of one of the circles.

$$A = \pi r^2$$

$$A = \pi (20)^2 \approx 1256.64 \text{ ft}^2$$

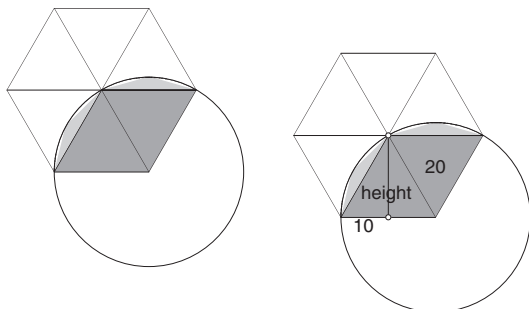
Each angle of the hexagon is 120° . Find the area of the sector of the circle.



The hexagon is composed of six equilateral triangles. The area of half the sector shown is $\frac{60}{360}$ or $\frac{1}{6}$ of the area of the circle. Thus, to determine the area of the sector, multiply the area of the circle by one-sixth.

$$A = \frac{1}{6}(1256.64) = 209.44 \text{ ft}^2$$

In order to find the area of the segments of the circle that form a petal, the area of the triangle must be subtracted from the area of the one-sixth sector of the circle.



Determine the height of the triangle.

$$h^2 + 10^2 = 20^2$$

$$h^2 = 300$$

$$h = 10\sqrt{3} \approx 17.32 \text{ ft}$$

The area of the triangle is

$$A = \frac{1}{2}bh = \frac{1}{2}(20)(17.32) = 173.2 \text{ ft}^2$$

The area of one-half of a petal is the area of the sector minus the area of the triangle.

$$209.44 - 173.2 = 36.24 \text{ ft}^2$$

The area of one petal is $2(36.24) \text{ ft}^2 = 72.48 \text{ ft}^2$

Multiplying by 6 yields the total area of the petals.

$$\text{Total area} = 6(72.48) = 434.88 \text{ ft}^2$$

(G.8) Congruence and the geometry of size.

The student uses tools to determine measurements of geometric figures and extends measurement concepts to find perimeter, area, and volume in problem situations.

The student is expected to:

(C) derive, extend, and use the Pythagorean Theorem; and

Connections to TAKS:

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

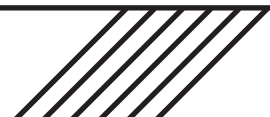
Extension Questions:

- If the length of the hexagon side is doubled, how is the area affected?

If the length is doubled, then the area is multiplied by a factor of two squared.

- Is it possible to use other regular polygons to create flowers using the same process?

What makes this activity possible is the fact that the side of the hexagon becomes the radius of the circle. The hexagon is the only one for which this relationship is true.



Student Work Sample

The next page shows the work of an individual student. The teacher required her students to give a verbal description of the process used to solve the problem. The second page shows the student's computations. The work exemplifies the following criteria:

- Communicates clear, detailed, and organized solution strategy.

The student provides a step-by-step description of the steps. He gives a reason for each step.

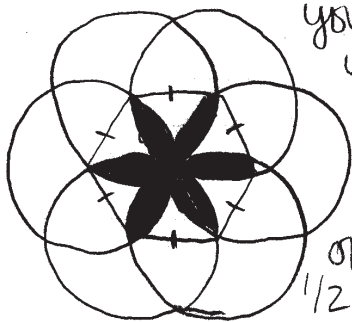
- Demonstrated geometric concepts, process, and skills.

She used the Pythagorean Theorem. She showed how to get the area of the hexagon and the area of a circle. She showed how to find the area of the petal region by subtracting the area of the hexagon from the area of the circle.

Flower

A landscaping company has a flower design. The petals are constructed by drawing circles from the vertices to the center of a regular hexagon, with a radius of 20 feet. The company must know the area of the petals so they will know how many flowers to buy. Find the area of the petals.

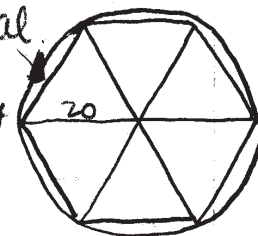
To start off the problem, you know that the hexagon is regular, which means that all of the sides are equal.



you also know that if you draw a circle around a regular hexagon, to where it touches each of the 6 corners, the space outside of the hexagon is equal to $\frac{1}{2}$ of a petal.

$\frac{1}{2}$ of a single petal.

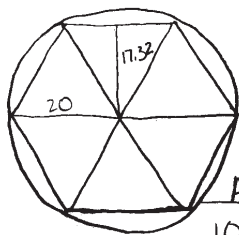
you find the area of the hexagon, and then subtract that from the area of the circle that you drew.



$(10\sqrt{300})/2 \cdot 12 = 1039.23$. Then the circle $\pi 20^2$ so subtract, $\pi 20^2 - (10\sqrt{300})/2 \cdot 12$ and you get 217.41. you know that's $\frac{1}{2}$ of all of the petals, so you multiply that by 2 to get 434.81 ft^2 .

→

This problem used the pythagorean theorem, so that you could figure out the area of the hexagon. Also, you used πr^2 to find the area of the circle. you then used subtraction and addition to find the total area of the petals.



Work

Height of 1 hexagon

$$10^2 + x^2 = 20^2$$

$$x^2 = 300$$

$$x = \sqrt{300} \text{ or } 17.32$$

Area of all Hexagons

$$10(\sqrt{300}) = 173.21 / 2 = 86.60(12) = 1039.23$$

Area of circle

$$\pi 20^2 = 1256.64$$

$$1256.64$$

$$- 1039.23$$

$$\hline 217.41$$

$$\times \quad 2$$

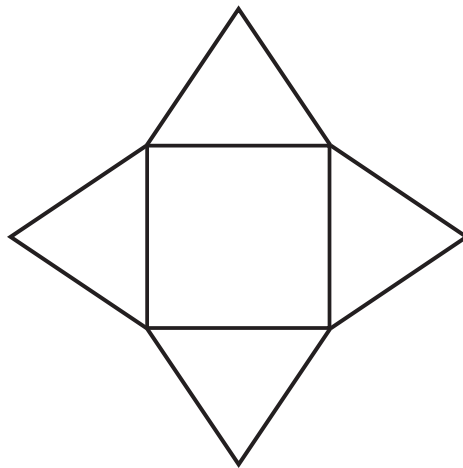
Area of flower → 434.81

Area of flower petals: 434.81 ft²

Great Pyramid

King Khufu, the second pharaoh of the fourth Egyptian dynasty, commissioned the building of the Great Pyramid of Giza. This monument was built of solid stone and was completed in just under 30 years, around 2560 BCE. It presides over the plateau of Giza on the outskirts of Cairo, and is the last survivor of the Seven Wonders of the World. At 481 feet high, the Great Pyramid stood as the tallest structure in the world for more than 4,000 years. The base of the Great Pyramid was a square with each side measuring 756 feet.

1. If the average stone used in the construction was a cube measuring 3.4 feet on a side, approximately how many stones were used to build this solid monument? Justify your answer.
2. If you could unfold the pyramid like the figure below, how many acres of land would it cover? (1 acre = 43,560 square feet)





Notes

Materials:

One graphing calculator per student

Geometry TEKS Focus:

(G.6) **Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional geometric figures and related two-dimensional representations and uses these representations to solve problems.

The student is expected to:

(B) use nets to represent and construct three-dimensional geometric figures; and

(G.8) **Congruence and the geometry of size.** The student uses tools to determine measurements of geometric figures and extends measurement concepts to find perimeter, area, and volume in problem situations.

The student is expected to:

(D) find surface areas and volumes of prisms, pyramids, spheres, cones, cylinders, and composites of these figures in problem situations.

Teacher Notes

Scaffolding Questions:

- What shape is the solid figure? How do you know?
- What formula do you need in order to find the number of stones?
- Explain what measurements you will use in order to determine the volume of the pyramid.
- What are the component parts of the net of the pyramid?
- Describe how you could find the surface area of the pyramid.

Sample Solutions:

1. Find the volume of a stone and the pyramid.

Stone

$$V = s^3$$

$$V = 3.4^3$$

$$V = 39.304 \text{ ft}^3$$

Pyramid

$$V = \frac{1}{3}Bh$$

$$V = \frac{1}{3}(756)^2(481)$$

$$V = 91,636,272 \text{ ft}^3$$

$$\text{Number of stones} = \frac{V_{\text{pyramid}}}{V_{\text{stone}}} = \frac{91636272}{39.304} \approx 2,331,474.456$$

If whole number of stones are required, the number of stones would be 2,332,475.

2. Find the area of component parts.

Square

$$A = s^2$$

$$A = 756^2$$

$$A = 571536 \text{ ft}^2$$

Triangle

Find the slant height of the pyramid; this is the hypotenuse of the cross section of the pyramid.

$$a^2 + b^2 = c^2$$

$$481^2 + \left(\frac{1}{2} \cdot 756\right)^2 = (\text{slant height})^2$$

$$\sqrt{374245} = \text{slant height}$$

$$611.76 \text{ ft} \approx \text{slant height}$$

Now find the area of a side.

$$A = \frac{1}{2}bh$$

$$A = \frac{1}{2} \cdot 756 \cdot 611.76$$

$$A \approx 231245.28 \text{ ft}^2$$

The four sides of the pyramid are congruent. The total surface area of the pyramid is the area of the square base plus the area of the four sides.

$$\begin{aligned} \text{Total area} &\approx 571536 + 4(231245.28) \\ &\approx 1496517.12 \text{ ft}^2 \end{aligned}$$

$$\text{Convert to acres: } 1496517.12 \text{ ft}^2 \left(\frac{1 \text{ acre}}{43560 \text{ ft}^2} \right) \approx 34 \text{ acres}$$

Extension Questions:

- If all the dimensions of the pyramid were multiplied by 4, how would the volume of the pyramid be changed?

If a side is multiplied by a factor of 4, the volume is multiplied by a factor of 4^3 .

- If you want to have the net cover twice the surface area, how would the sides of the pyramid need to be changed?

If a side is multiplied by a factor of k , the area is multiplied by a factor of k^2 . If the area is multiplied by a factor of h , the side is multiplied by a factor of \sqrt{h} .

Additional Geometry TEKS:

(G.8) **Congruence and the geometry of size.** The student uses tools to determine measurements of geometric figures and extends measurement concepts to find perimeter, area, and volume in problem situations.

The student is expected to:

(C) derive, extend, and use the Pythagorean Theorem; and

(G.9) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student is expected to:

(D) analyze the characteristics of polyhedra and other three-dimensional figures and their component parts based on explorations and concrete models.

Connections to TAKS:

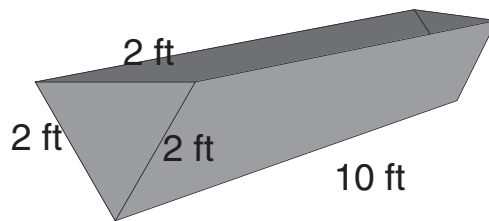
Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Walter and Juanita's Water Troughs

You have been hired as chief mathematician by a company named Walter and Juanita's Water Troughs. This company builds water troughs for various agricultural uses. The company has one design (see figure 1.). Your job is to perform mathematical analysis for the owners.

figure 1



1. A customer would like to know what the depth of the water is (in inches) if the trough has only 32 gallons in it.
2. The interior of the troughs must be coated with a sealant in order to hold water. One container of sealant covers 400 square feet. Will one container of sealant be enough to seal ten troughs? Why or why not?
3. Walter and Juanita would like to explore some minor modifications of their original design. They would like to know which change will produce a water trough that would hold more water—adding one foot to the length of the trough, making it 11 feet long, or adding three inches to each side of the triangular bases, making them 2 feet 3 inches on each side (see figures 2 and 3). Justify your answer.

figure 2

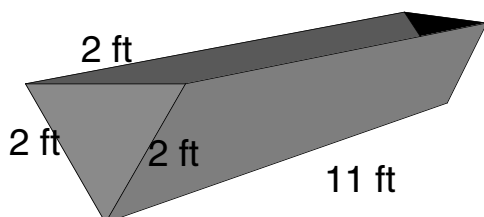
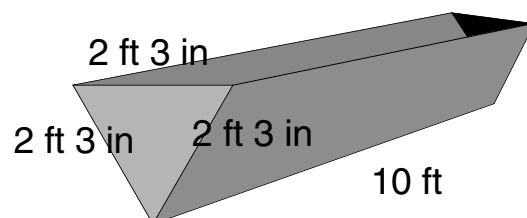


figure 3



Notes

Materials:

One calculator and straightedge per student

Geometry TEKS Focus:

(G.5) **Geometric patterns.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to:

(D) identify and apply patterns from right triangles to solve meaningful problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

(G.8) Congruence and the

geometry of size. The student uses tools to determine measurements of geometric figures and extends measurement concepts to find perimeter, area, and volume in problem situations.

The student is expected to:

(D) find surface areas and volumes of prisms, pyramids, spheres, cones, cylinders, and composites of these figures in problem situations.

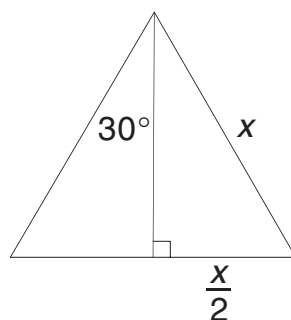
Teacher Notes

Scaffolding Questions:

- What important components of the trough are needed to solve the problems?
- What is your prediction for which trough will hold the most water in number 3? Justify your answer.

Sample Solutions:

1. To solve this problem, the area of the triangular base must be found in terms of an unknown side. If water is poured into the trough, the height of the water is a function of the side of the triangle. Consider the end of the trough that is an equilateral triangle.



Using 30-60-90 special right triangle properties or the Pythagorean Theorem, the altitude is $\frac{x\sqrt{3}}{2}$ ft.

Area of the base can be found by:

$$\text{Area} = \frac{1}{2}(x)\left(\frac{x\sqrt{3}}{2}\right)$$

$$\text{Area} = \frac{x^2\sqrt{3}}{4} \text{ ft}^2$$

The desired volume is given in gallons and must be converted to cubic feet because the measurements are given in feet.

$$32 \text{ gal.} \cdot \frac{1 \text{ ft}^3}{7.48 \text{ gal}} \approx 4.28 \text{ ft}^3$$

The volume of the tank is the area of the triangular base times the length of the tank.

$$\left(\frac{x^2\sqrt{3}}{4}\right)10 = 4.28 \text{ ft}^3$$

$$(x^2\sqrt{3})10 \approx 17.12$$

$$x \approx \pm\sqrt{0.988} \approx 0.994 \text{ ft}$$

The side length is 0.994 ft.

The depth of the water is the altitude of the base.

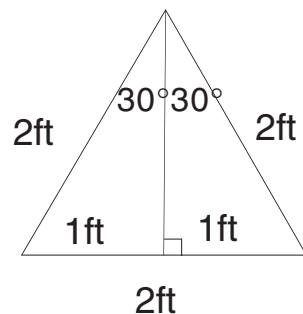
$$\text{altitude} = \frac{0.994\sqrt{3}}{2}$$

$$\text{altitude} \approx 0.861 \text{ ft}$$

$$\text{depth of water} \approx 10.3 \text{ in}$$

2. Find the interior surface area of one trough and multiply by 10.

Figure 2: First, find the area of the base (the ends of the trough). In order to do this, find the altitude of the triangle. The base is an equilateral triangle, so the altitude bisects the side and the intercepted angle. The angles of an equilateral triangle are all 60° .



Use the Pythagorean Theorem to find the altitude.

$$a^2 + b^2 = c^2$$

$$a^2 + 1^2 = 2^2$$

$$a^2 + 1 = 4$$

$$a^2 = 3$$

$$a = \pm\sqrt{3}$$

Additional Geometry TEKS:

(G.8) **Congruence and the geometry of size.** The student uses tools to determine measurements of geometric figures and extends measurement concepts to find perimeter, area, and volume in problem situations.

The student is expected to:

(A) find areas of regular polygons, circles, and composite figures;

(C) derive, extend, and use the Pythagorean Theorem; and

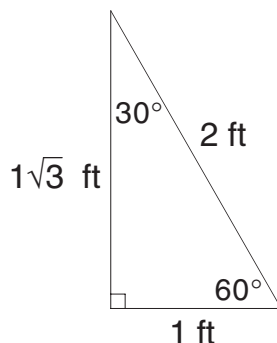
Connections to TAKS:

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

The altitude must be positive.

$$a = \sqrt{3}$$

Another approach to determining the altitude is to use 30-60-90 special right triangle properties. The hypotenuse is twice the shortest side. The side opposite the 60-degree angle is the shorter leg times $\sqrt{3}$.



The altitude is $1\sqrt{3} \approx 1.73$ ft.

The area of the triangular base may be found using the altitude and the base of the triangle.

$$\text{Area} = \frac{1}{2}bh = \frac{1}{2}(2)(\sqrt{3}) = \sqrt{3} \approx 1.73 \text{ ft}^2$$

The surface area is the sum of the areas of the 2 ends and the 2 sides.

$$2\sqrt{3} + 2(20) \approx 43.64 \text{ ft}^2$$

The surface area of ten troughs is 10 times the surface area of one trough.

$$(10)43.46 = 434.6 \text{ ft}^2$$

Since the gallon of paint covers 400 ft^2 , there will not be enough sealant to seal ten troughs.

3. The volume of a right prism is the area of the base times the height.

The area of the base for the first case was found in problem 2.

$$\text{Area} = \frac{1}{2}bh = \frac{1}{2}(2)(\sqrt{3}) = \sqrt{3} \approx 1.73 \text{ ft}^2$$

$$V = Bh = \sqrt{3}(11) \approx 1.73(11) \approx 19.05 \text{ ft}^3$$

For the second case, find the altitude of the right triangle base with side of 2.25 feet and the shorter leg, one-half of 2.25 or 1.125. Use the Pythagorean Theorem.

$$a^2 + b^2 = c^2$$

$$a^2 + (1.125)^2 = (2.25)^2$$

$$a^2 \approx 5.0625 - 1.266$$

$$a \approx \sqrt{3.8}$$

$$a \approx 1.95 \text{ ft}$$

Find the area of the base.

$$A = \frac{1}{2}bh$$

$$A \approx \frac{1}{2}(2.25)(1.95)$$

$$A \approx 2.19 \text{ ft}^2$$

Find the volume.

$$V = Bh$$

$$V \approx (2.19)(10)$$

$$V \approx 21.9 \text{ ft}^3$$

Adding 3 inches to each side of the base will produce a greater increase in volume than adding a foot to the distance between the bases.

Extension Question:

- A new tank is designed in the shape of a hemisphere (half of a sphere) to hold the same volume of water as the tank in figure 1. What is the depth of the water if the tank is full?

The area of the base (the ends of the trough) is the same as the area for figure 2. The area is $\sqrt{3} \approx 1.73 \text{ ft}^2$. The volume of a right prism is the area of the base times the height.

$$V = Bh = \sqrt{3}(10) \approx 17.3 \text{ ft}^3$$

Volume of a sphere = $\frac{4}{3}\pi r^3$, but we want only $\frac{1}{2}$ of this amount.

$$V = \frac{1}{2} \left(\frac{4}{3} \pi r^3 \right)$$

$$17.3 = \frac{1}{2} \left(\frac{4}{3} \pi r^3 \right)$$

$$17.3 = \frac{2}{3} \pi r^3$$

$$\frac{3}{2} \cdot 17.3 = \frac{3}{2} \cdot \frac{2}{3} \pi r^3$$

$$\frac{25.95}{\pi} = \frac{\pi}{\pi} r^3$$

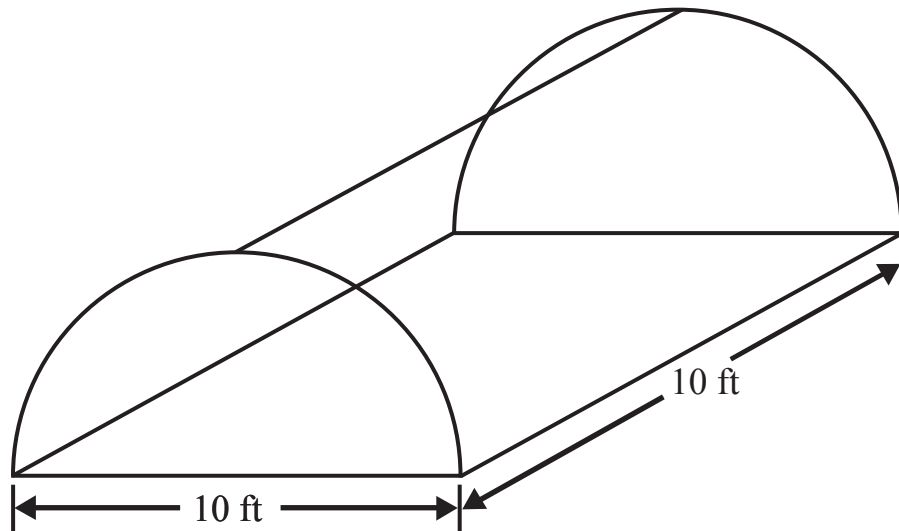
$$\sqrt[3]{8.26} = \sqrt[3]{r^3}$$

$$2.02 \approx r$$

The depth is approximately 2.02 feet.

Greenhouse

The backyard greenhouse in the figure below uses plastic tubing for framing and plastic sheeting for wall covering. The end walls are semicircles, and the greenhouse is built to the dimensions in the figure below. All walls and the floor are covered by the plastic sheeting. The door is formed by cutting a slit in one of the end walls.



1. How many square feet of plastic sheeting will it take to cover the top, sides, and floor of the greenhouse?
2. What is the volume of the greenhouse?



Notes

Materials:

One graphing calculator per student

Geometry TEKS Focus:

(G.8) **Congruence and the geometry of size.** The student uses tools to determine measurements of geometric figures and extends measurement concepts to find perimeter, area, and volume in problem situations.

The student is expected to:

(A) find areas of regular polygons, circles, and composite figures;

(D) find surface areas and volumes of prisms, pyramids, spheres, cones, cylinders, and composites of these figures in problem situations.

Teacher Notes

Scaffolding Questions:

- Describe the shape of the greenhouse.
- What dimensions do you need to determine the surface area?

Sample Solutions:

1. The ends of the greenhouse are semicircles that have a diameter of ten. Area of each end:

$$A = \pi r^2$$

$$A = \pi(5)^2 \approx 78.54$$

The area of one-half of the circle is $A \approx 78.54 \cdot \frac{1}{2} \approx 39.27 \text{ ft}^2$

Area of side:

Since the greenhouse is half of a cylinder, we can unwrap the sides, find the area of the rectangle, and divide by 2. One dimension of the rectangle is the circumference, and the other is 10.

$$C = \pi d$$

$$C = \pi 10$$

$$C \approx 31.4 \text{ ft}$$

Find the area.

$$\text{Surface area of the cylinder} \approx 31.4 \cdot 10 \approx 314 \text{ ft}^2$$

$$\begin{aligned} \text{Surface area of half of the cylinder} &\approx 314 \text{ ft}^2 \cdot \frac{1}{2} \\ &\approx 157 \text{ ft}^2 \end{aligned}$$

The floor is the area of the square.

$$A = 10 \cdot 10$$

$$A = 100 \text{ ft}^2$$

Total square feet of plastic needed is

$$39.27 + 39.27 + 157 + 100 = 335.54 \text{ ft}^2.$$

2. The volume of the greenhouse is the area of the base multiplied by the distance between the bases. The base is a semicircle with a diameter of 10.

$$A = \frac{1}{2} \pi r^2 \text{ because we only need half of the circle.}$$

$$A = \frac{1}{2} \pi (5)^2$$

$$A \approx 39.27 \text{ ft}^2$$

The volume of the greenhouse may be computed using the formula.

$$V = Bh$$

$$V = 39.27 \cdot 10$$

$$V \approx 392.7 \text{ ft}^3$$

Extension Question:

- What would the dimensions of a new square floor greenhouse have to be in order to double the volume of the greenhouse in this problem?

Let the side length be x . The radius of the semicircular ends would be expressed as $\frac{x}{2}$.

The area of the base is one-half of the area of a circle with radius $\frac{x}{2}$.

$$A = \frac{1}{2} \pi \left(\frac{1}{2} x \right)^2$$

$$A = \frac{1}{8} \pi x^2$$

The volume of the greenhouse is the area of the base (the area of the semicircle) times the length, x .

Additional Geometry TEKS:

(G.8) **Congruence and the geometry of size.** The student uses tools to determine measurements of geometric figures and extends measurement concepts to find perimeter, area, and volume in problem situations.

The student is expected to:

(B) find areas of sectors and arc lengths of circles using proportional reasoning;

Connections to TAKS:

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

The rule for the volume as a function of the side is:

$$V = \frac{1}{8} \pi x^2 \cdot x$$

The volume must two double the original volume or two times 392.7 ft^3 . The volume must be 785.4 ft^3 .

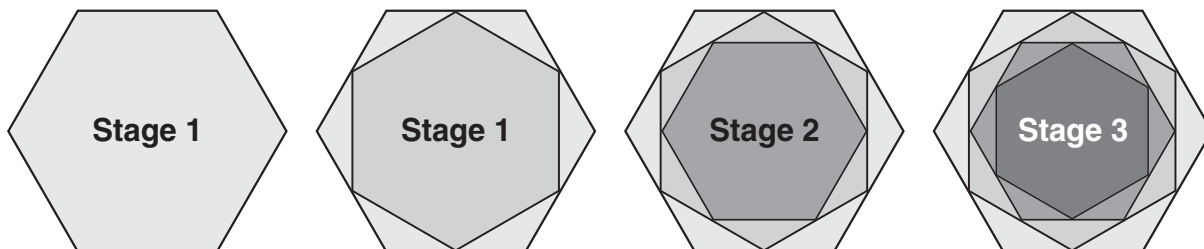
$$785.4 = \frac{1}{8} \pi x^3$$

$$x \approx 12.6 \text{ ft}$$

The length of the side of the base must be about 12.6 feet.

Nesting Hexagons

The first four stages of nested hexagons are shown below. Connecting the midpoints of the sides of the previous stage creates each successive stage. The side of the hexagon in Stage 0 measures 3 units.



1. Create a function rule to find the area of the innermost hexagon of any stage.
2. Find the area of the innermost hexagon in stage 10.
3. What is the domain of your function rule?
4. What is the range of your function rule?



Notes

Materials:

One graphing calculator and straightedge per student

Geometry TEKS Focus:

(G.5) **Geometric patterns.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to:

(A) use numeric and geometric patterns to develop algebraic expressions representing geometric properties;

(G.8) **Congruence and the geometry of size.** The student uses tools to determine measurements of geometric figures and extends measurement concepts to find perimeter, area, and volume in problem situations.

The student is expected to:

(A) find areas of regular polygons, circles, and composite figures;

Additional Geometry TEKS:

(G.5) **Geometric patterns.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to:

(B) use numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles;

Teacher Notes

Note: It may prove beneficial to create a geometry software sketch of “Nesting Hexagons” once students have performed the calculations for Stages 1 and 2. Students can explore whether the size of the hexagon has any impact on the relationship between the stages. They can also use the measurements to help determine a function rule for this situation.

It may also prove beneficial to model the organization of data in a table such as the one in the sample solutions. Encourage students to round area values to the nearest hundredth.

Scaffolding Questions:

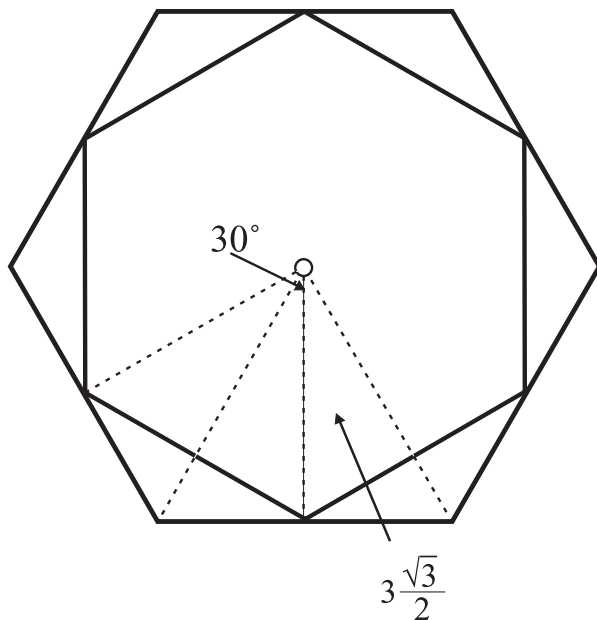
If students have created a table, the following questions may be asked:

- Is there a common difference between the terms in the area column of the table?
- Is there a common second difference between the terms in the table?
- Is there a common ratio between the terms in the table?
- How is each stage related to the previous stage?

Sample Solutions:

1. The perimeter is 18. Use $A = \frac{1}{2}ap$ to find the area of the regular hexagon. Each interior angle of a regular hexagon is 120° . The segments from each vertex to the center bisect the interior angles forming six equilateral triangles. Therefore the radius and the sides of the hexagon have the same measure.

Using 30-60-90 right triangle properties, the length of the apothem is found by dividing the hypotenuse, or in this case the radius, by 2 and multiplying by $\sqrt{3}$. The length of the apothem for stage 0 is $3\frac{\sqrt{3}}{2}$, and the length of the radius is 3. The perimeter, p , is 6 times 3 units, or 18 units.



$$A = \frac{1}{2} ap$$

$$A = \frac{1}{2} \left(3 \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \right) \left(18 \frac{\sqrt{3}}{2} \right)$$

$$A = \frac{1}{2} \cdot 3 \cdot 18 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$$

$$A = 27 \frac{\sqrt{3}}{2} \cdot \frac{3}{4} \approx 17.54 \text{ square units}$$

Stage	Apothem	Radius	Side Length	Perimeter	Area	Area rounded to the nearest hundredth
0	$3 \frac{\sqrt{3}}{2}$	3	3	18	$27 \frac{\sqrt{3}}{2}$	23.38

The apothem of the stage 0 hexagon is now the radius of the stage 1 hexagon. The length of the side of the stage 1 hexagon is $3 \frac{\sqrt{3}}{2}$.

Using 30-60-90 right triangle properties, the length of the side is $3 \frac{\sqrt{3}}{2}$, and the length of the apothem is $3 \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$.

(D) identify and apply patterns from right triangles to solve meaningful problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

Connections to TAKS:

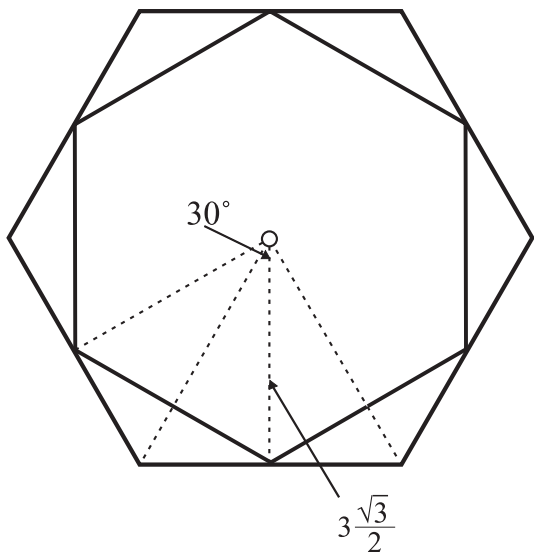
Objective 1: The student will describe functional relationships in a variety of ways.

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Note: It may prove beneficial to create a geometry software sketch of “Nesting Hexagons” once students have performed the calculations for Stages 1 and 2. Students can explore whether the size of the hexagon has any impact on the relationship between the stages. They can also use the measurements to help determine a function rule for this situation.

It may also prove beneficial to model the organization of data in a table such as the one in the possible solution strategies. Encourage students to round area values to the nearest hundredth.



$$A = \frac{1}{2}ap$$

$$A = \frac{1}{2} \left(3 \frac{\sqrt{3}}{2} \right) (18)$$

$$A \approx \frac{1}{2} \cdot 3 \cdot 18 \cdot \frac{\sqrt{3}}{2}$$

$$A = 27 \frac{\sqrt{3}}{2} \text{ or}$$

$$A \approx 23.38 \text{ square units}$$

Stage	Apothem	Radius	Side Length	Perimeter	Area	Area rounded to the nearest hundredth
0	$3 \frac{\sqrt{3}}{2}$	3	3	18	$27 \frac{\sqrt{3}}{2}$	23.38
1	$3 \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$	$3 \frac{\sqrt{3}}{2}$	$3 \frac{\sqrt{3}}{2}$	$18 \frac{\sqrt{3}}{2}$	$27 \frac{\sqrt{3}}{2} \cdot \frac{3}{4}$	17.54

A pattern emerges when the rest of the table is completed. In each new stage the side length, radius, apothem, and perimeter change by a factor of $\frac{\sqrt{3}}{2}$ times the amount in the previous stage. The area in the new stage changes by a factor of $\frac{3}{4}$ times the amount in the previous stage.

Stage	Apothem	Radius	Side Length	Perimeter	Area	Area rounded to the nearest hundredth
0	$3\frac{\sqrt{3}}{2}$	3	3	18	$27\frac{\sqrt{3}}{2}$	23.38
1	$3\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$	$3\frac{\sqrt{3}}{2}$	$3\frac{\sqrt{3}}{2}$	$18\frac{\sqrt{3}}{2}$	$27\frac{\sqrt{3}}{2} \cdot \frac{3}{4}$	17.54
2	$3\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$	$3\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$	$3\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$	$18\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$	$27\frac{\sqrt{3}}{2} \cdot \frac{3}{4} \cdot \frac{3}{4}$	13.15
3	$3\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$	$3\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$	$3\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$	$18\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$	$27\frac{\sqrt{3}}{2} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4}$	9.86

A recursive process of repeated multiplication can be used to generate the function rule for the area.

Stage	Area
0	$27\frac{\sqrt{3}}{2}$
1	$27\frac{\sqrt{3}}{2} \cdot \frac{3}{4}$
2	$27\frac{\sqrt{3}}{2} \cdot \frac{3}{4} \cdot \frac{3}{4}$
3	$27\frac{\sqrt{3}}{2} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4}$
4	$27\frac{\sqrt{3}}{2} \left(\frac{3}{4}\right)^4$
x	$27\frac{\sqrt{3}}{2} \left(\frac{3}{4}\right)^x$

Another approach is to look for patterns in the area values themselves. It will facilitate the development of a pattern if the values are rounded to the nearest hundredth. If the difference between each term is found, no pattern emerges. If the difference between each of those differences is found, no pattern emerges. However, if the ratios of the terms are found, a common ratio emerges. This means that each successive term can be found by multiplying the previous term by this common ratio. This leads to the generation of the exponential rule for this situation.

Stage	Process	Area	1 st differences	2 nd differences	common ratio
0	23.38 or $23.38 = 23.38 (0.75)^0$	23.38			
1	$23.38 (0.75)$ or $23.38 (0.75) = 23.38 (0.75)^1$	17.54	5.84		$\frac{17.54}{23.38} \approx 0.75$
2	$17.54 (0.75)$ or $23.38 (0.75)(0.75) = 23.38 (0.75)^2$	13.15	4.39	1.45	$\frac{13.15}{17.54} \approx 0.75$
3	$13.15 (0.75)$ or $23.38(0.75)(0.75)(0.75) = 23.38 (0.75)^3$	9.86	3.29	1.1	$\frac{9.86}{13.15} \approx 0.75$
Description	The area of the current stage hexagon is the area of the stage 0 hexagon multiplied by 0.75 raised to the current stage number.				
x	$23.38 (0.75)^x$	A			

$$A = 27 \frac{\sqrt{3}}{2} \left(\frac{3}{4}\right)^x$$

2. $A = 27 \frac{\sqrt{3}}{2} \left(\frac{3}{4}\right)^{10} \approx 1.32$

3. Possible notation for the domain of the function might include one of the following: all real numbers, $-\infty < x < +\infty$, or $(-\infty, +\infty)$.

The domain for the problem situation would be non-negative integers.

4. Possible notation for the range of the function might include one of the following: real numbers greater than 0, $0 < y < +\infty$, or $(0, +\infty)$.

The range for the problem situation is more complicated. The range is the specific set of numbers corresponding to the domain values for the problem situation. For example,

$$\left\{ \frac{27\sqrt{3}}{2}, \frac{81\sqrt{3}}{8}, \frac{243\sqrt{3}}{32}, \dots \right\}$$

Extension Questions:

- What is the relationship between the ratio of side lengths $\frac{\text{stage}(n)}{\text{stage}(n-1)}$ and the ratio of perimeters? How is this number related to the ratio of areas?

The ratio of side lengths and the ratios of perimeters are both $\frac{\sqrt{3}}{2}$. This ratio squared is the ratio of the areas.

- What is the largest possible area in this problem? Justify your answer.

The largest possible area is $27\frac{\sqrt{3}}{2}$ or 23.38.

The area of each successive stage is $\frac{3}{4}$ times the area of the hexagon of the previous stage, so the area is always smaller than the area of the hexagon of stage 0.

- What is the smallest possible area in this problem? Justify your answer.

There will never be a smallest area. Any area thought to be the smallest area can be multiplied by $\frac{3}{4}$ according to our rule in order to find a new stage with a smaller area. The area will approach 0 as the stage number gets very large but will never actually be 0.

Chapter 5:
*Solids
and Nets*

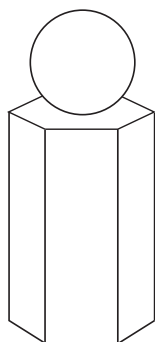


Introduction

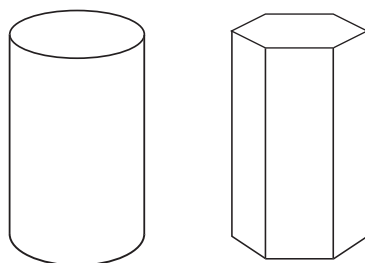
The performance tasks in this chapter require students to analyze relationships between two- and three-dimensional objects and use these relationships to solve problems.

Perfume Packaging

Gina would like to package her newest fragrance, *Persuasive*, in an eye-catching yet cost-efficient box. The *Persuasive* perfume bottle is in the shape of a regular hexagonal prism 10 centimeters high. Each base edge of the prism is 3 centimeters. The cap is a sphere with a radius of 1.5 centimeters.



Gina would like to compare the cost of the proposed box designs shown below. It is important that each bottle fit tightly in the box to avoid movement during shipping, but the box must be large enough to allow people to get the bottle in and out of it. Thus, the box should only be 0.5 cm taller and 0.5 cm wider than the height and widest portion of the base of the bottle.



1. Determine the measurements needed for each dimension on the two boxes. Explain how you determined these dimensions.
2. Sketch the net for each package design, showing how you would make each box from a single sheet of cardboard. (Do not include the flaps needed to glue the box.)
3. Calculate the amount of cardboard used for each design. Which box will be more cost-efficient?

Notes

Materials:

One graphing calculator per student

Geometry TEKS Focus:

(G.6) **Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional geometric figures and related two-dimensional representations and uses these representations to solve problems.

The student is expected to:

(B) use nets to represent and construct three-dimensional geometric figures; and

(G.8) **Congruence and the geometry of size.** The student uses tools to determine measurements of geometric figures and extends measurement concepts to find perimeter, area, and volume in problem situations.

The student is expected to:

(A) find areas of regular polygons, circles, and composite figures;

(D) find surface areas and volumes of prisms, pyramids, spheres, cones, cylinders, and composites of these figures in problem situations.

Additional Geometry TEKS:

(G.5) **Geometric patterns.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to:

(D) identify and apply patterns from right triangles to solve

Teacher Notes

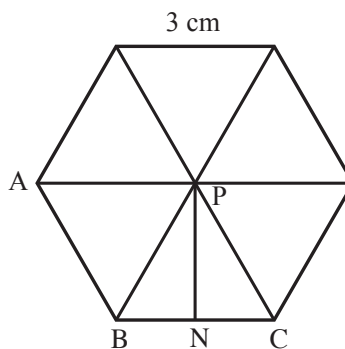
Scaffolding Questions:

- What is the total height of the perfume bottle and the cap?
- Describe the shape of the sides of each box.
- Describe the bases of each box.
- How will the dimensions of the perfume bottle compare to the dimensions of the box?

Sample Solutions:

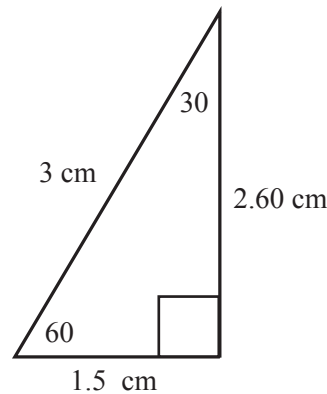
1. The total height of the perfume bottle and cap is 13 centimeters.

The base of the bottle is a regular hexagon with sides of 3 centimeters. A sketch of the base is shown below. The diagonals of the regular hexagon intersect in the center of the hexagon. The segment connecting the center of the regular hexagon and a vertex is the radius, \overline{AP} or \overline{BP} , of the regular hexagon.

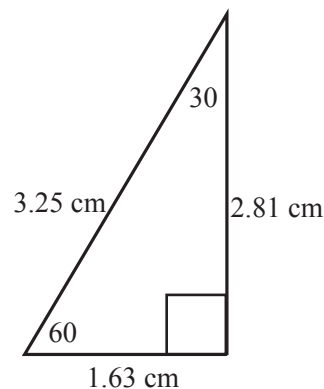


A regular hexagon has a radius equal to the length of the side of the regular hexagon. This can be established by drawing in the apothem of the hexagon and using a 30-60-90 triangle. The regular hexagon has an interior angle, $\angle ABC$, measuring 120 degrees. The radius of the hexagon, \overline{BP} , bisects this angle forming a 60-degree angle, $\angle CBP$. The apothem \overline{PN} forms a 90-degree angle with the side of the hexagon. The apothem also bisects the side of the hexagon; therefore, \overline{BN} measures 1.5 centimeters. The remaining angle of the triangle, $\angle BNP$, measures 30 degrees.

Using this information, it can be determined that the hypotenuse of the triangle is 3 centimeters. The hypotenuse of the triangle is also the radius of the hexagon. The following figure shows the measurements in the 30-60-90 triangle. Measurements are rounded to the nearest hundredth.



The required 0.5 cm allowance for packaging space must be considered. This changes the length of the diagonal of the hexagon to 6.5 cm. The diagonal is made up of 2 radii (shown as the hypotenuse in the triangle below). Therefore the actual measurement needed for the packaging must be adjusted on the triangle as follows:



The widest dimension of the base will be 6.5 centimeters. This dimension will be the same for the hexagon and the circular base.

The shorter distance across the hexagon (2 apothems, end to end) will be approximately 5.62 cm. (This amount is rounded.)

The measurement of the distance needed to go around the bottle will be the perimeter of the hexagon and the circumference of the circular base of the cylinder.

meaningful problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

(G.8) Congruence and the geometry of size. The student uses tools to determine measurements of geometric figures and extends measurement concepts to find perimeter, area, and volume in problem situations.

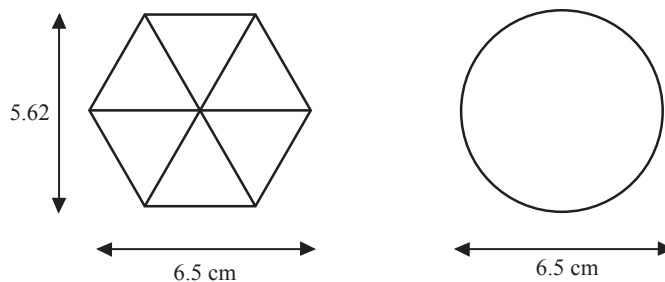
The student is expected to:

(B) find areas of sectors and arc lengths of circles using proportional reasoning;

Connections to TAKS:

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.



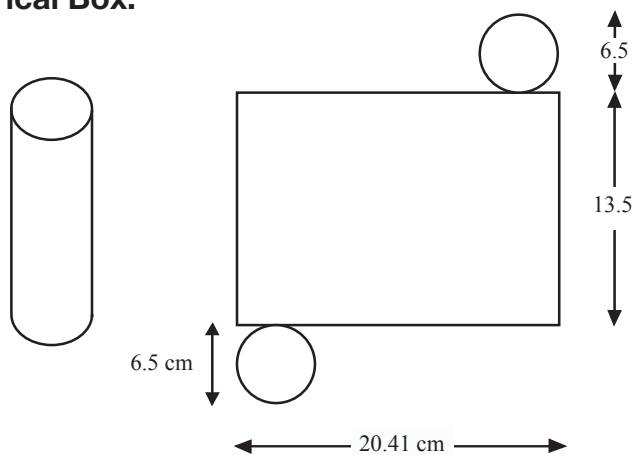
The perimeter of the regular hexagon is calculated by multiplying the length of one side by six. $P = 3.25 \text{ cm} \cdot 6 = 19.5 \text{ cm}$.

The circumference of the circular base is calculated using $C = 2\pi r$. The radius of the circle is 3.25 cm. So, $C = 2\pi \cdot (3.25 \text{ cm}) = 6.5\pi \text{ cm}$. Using 3.14 for π , the length of the circumference is:

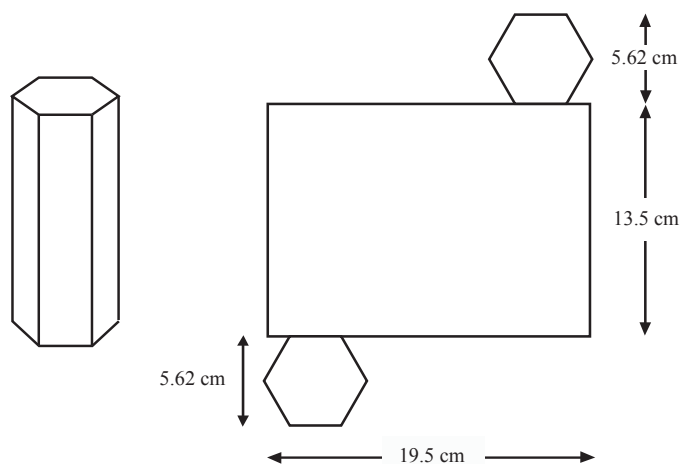
$$C = 6.5(3.14) \text{ cm} = 20.41 \text{ cm}.$$

- The sketch of the nets for the boxes is shown below along with the surface area of each net. The 0.5 cm allowance for space is included in the dimensions. This space allows for the bottle to slide in and out of the box with ease, yet provides minimum movement during shipping. Values are rounded to the nearest hundredth of a centimeter.

Cylindrical Box:



Hexagonal Box:



3. The net of the cylindrical box consists of two congruent circles and a rectangle.

The area of each circle is $A = \pi r^2$.

$$A = \pi (3.25 \text{ cm})^2$$

$$A = (3.14)(10.56 \text{ cm})$$

$$A = 33.16 \text{ cm}^2$$

The area of the rectangle is $A = bh$.

$$A = 20.41 \text{ cm} \cdot 13.5 \text{ cm}$$

$$A = 275.54 \text{ cm}^2$$

The total surface area of the cylinder is found by adding the areas of the circles and the rectangle: $33.16 \text{ cm}^2 + 33.16 \text{ cm}^2 + 275.54 \text{ cm}^2 = 341.86 \text{ cm}^2$

The net of the hexagonal box is comprised of two regular hexagons and a rectangle.

The area of the rectangle is $A = bh$.

$$A = 19.5 \text{ cm} \cdot 13.5 \text{ cm}$$

$$A = 263.25 \text{ cm}^2$$

The area of the regular hexagon is $A = \frac{1}{2}(ap)$.

We know that $a = 2.81 \text{ cm}$ and $p = 6(3.25 \text{ cm}) = 19.5 \text{ cm}$

$$A = (2.81 \text{ cm} \cdot 19.5 \text{ cm}) / 2$$

$$A = (54.8 \text{ cm}^2) / 2$$

$$A = 27.4 \text{ cm}^2$$

The total surface area of the hexagonal box is

$$A = 263.25 \text{ cm}^2 + 27.4 \text{ cm}^2 + 27.4 \text{ cm}^2$$

$$A = 318.05 \text{ cm}^2$$

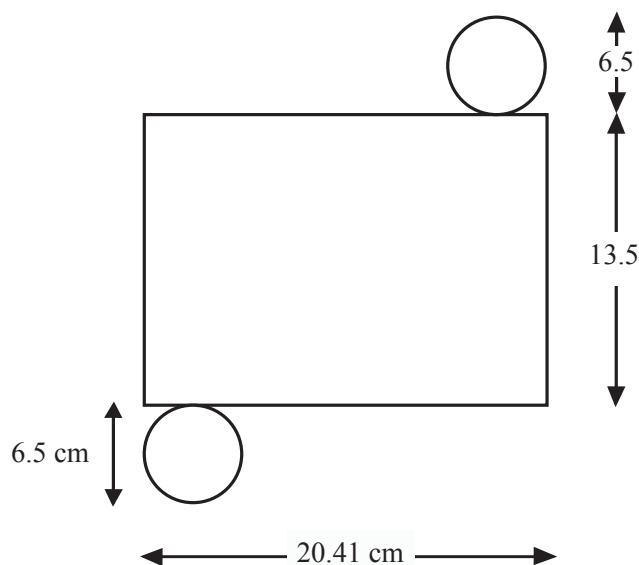
The hexagonal box has a smaller total surface area of 318.05 cm^2 , while the cylindrical box has a total surface area of 341.86 cm^2 . Gina should select the hexagonal box for the perfume because it is more cost-efficient.

Extension Question:

- Suppose the packaging manufacturer used sheets of cardboard 1 meter long and 1 meter wide. How many of each type of packaging could be made from a single sheet? Explain your answer in detail.

The area of the cardboard sheet is 1 square meter, or 10,000 square centimeters. You may not, however, divide this total by the total surface area of the box because the shapes, when placed on the sheet, will not use all of the space.

The cylindrical shape with tangent circle bases:



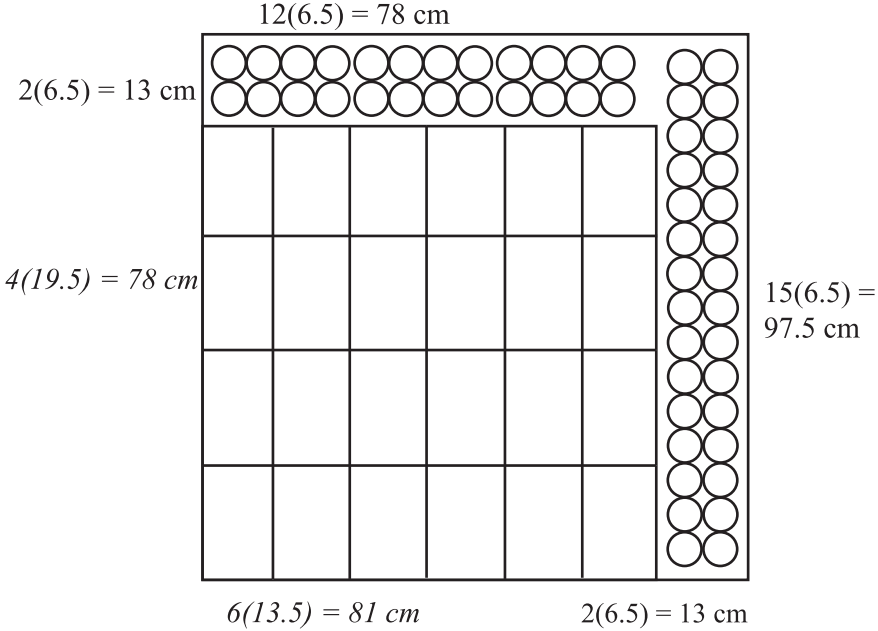
The total width is 20.41 cm, and the length is $6.5 + 6.5 + 13.5$, or 26.5 cm. If it is assumed that the shape must be placed with the tangent circles, the question is, “How many rectangles of this size will fit in a rectangle 100 cm by 100 cm?”

$$100 \div 26.5 \approx 3.77$$

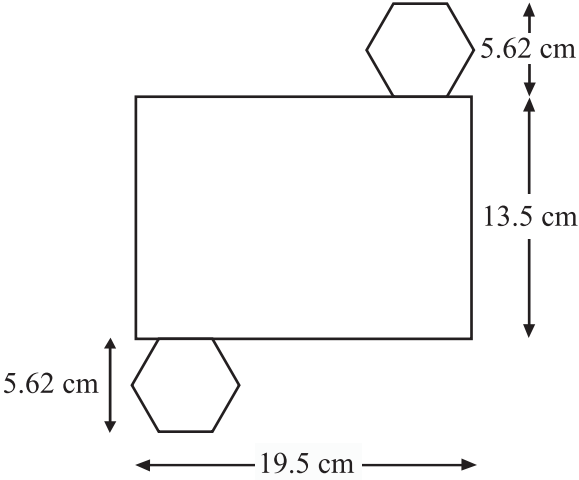
$$100 \div 20.41 \approx 4.9$$

An array of three rectangles by four rectangles would fit on the sheet. At least 12 shapes would fit on the sheet.

If the circles may be separated from the rectangle, more of the shapes could be cut out of the cardboard. One possible arrangement is shown below.



24 rectangles and 54 circles could be arranged on the square.

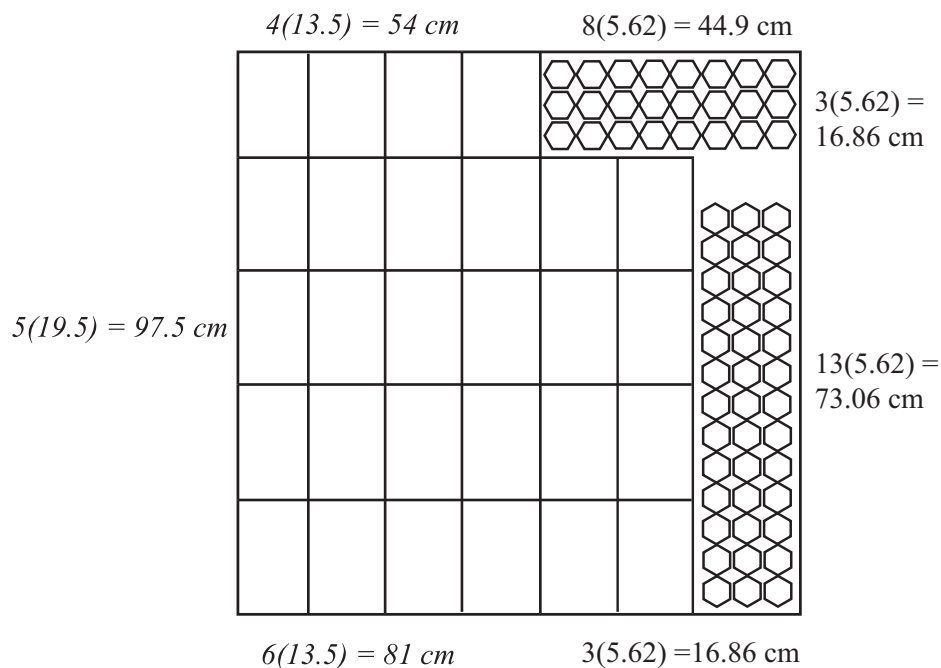


The total width is 19.5 cm, and the length is 5.62 + 5.62 + 13.5 or 24.74 cm. The question is, “How many rectangles of this size will fit in a rectangle 100 cm by 100 cm?”

$100 \div 19.5 \approx 5.128$
 $100 \div 24.74 \approx 4.04$

An array of five shapes by four rectangles would fit on the sheet. At least 20 shapes would fit on the sheet.

If the hexagons are not connected to the rectangles, more could be positioned on the cardboard. One possible arrangement is demonstrated in the diagram below.



Rectangles:

$4(5) = 20$

$2(4) = 8$

28 rectangles

Hexagons:

$13(3) = 39$

$8(3) = 24$

63 hexagons

Thus, 28 sides and 56 tops and bottoms will fit in the space.

Student Work

A group of four students created a poster of their solution to this problem. The work has been copied on the next three pages.

This work exemplifies all of the solution guide criteria. For example:

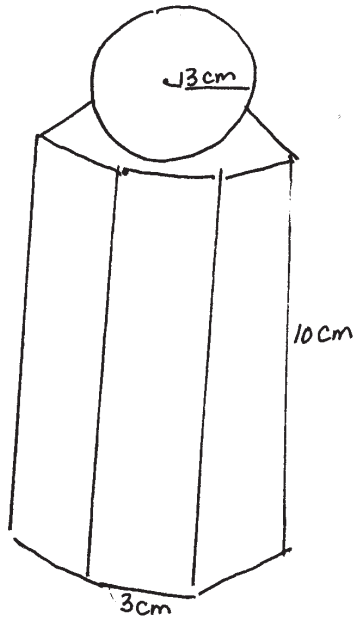
- Makes an appropriate and accurate representation of the problem using correctly labeled diagrams.

The students' diagrams of the solids and nets are correct. They have labeled the measurements of the perfume bottle, the containers, and the nets.

- Communicates clear, detailed, and organized solution strategy.

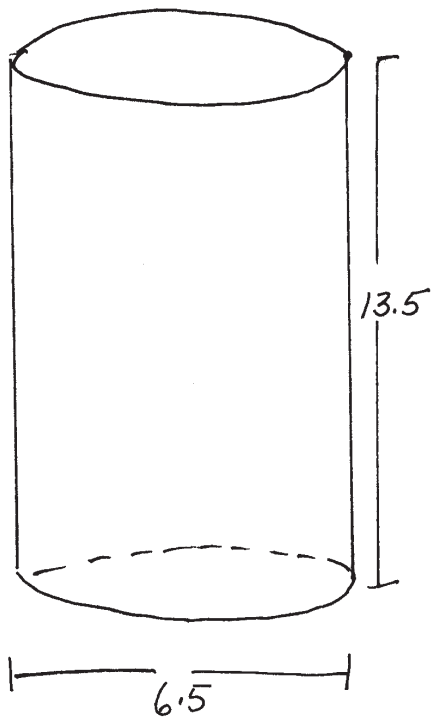
The students used the words "first," and "to find these measurements we did the following things." They described what they did and why. They listed the formulas used, what the variables in the formula represented, and what they were for this particular situation.

①

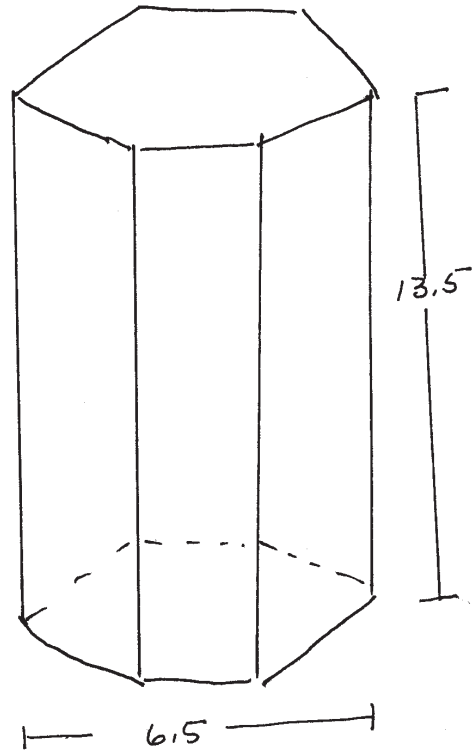


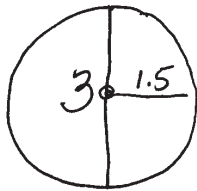
First, we were given the information that this perfume bottle was 10 cm high, each base edge was 3 cm wide, and the radius of the sphere was 1.5 cm. So we labeled accordingly.

Option #1



Option #2

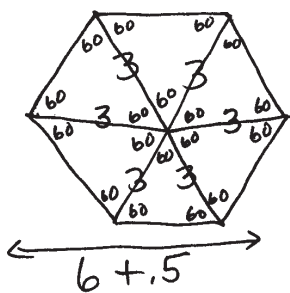




$$\begin{array}{r} +10 \\ 13 \\ +.5 \\ \hline 13.5 \end{array}$$

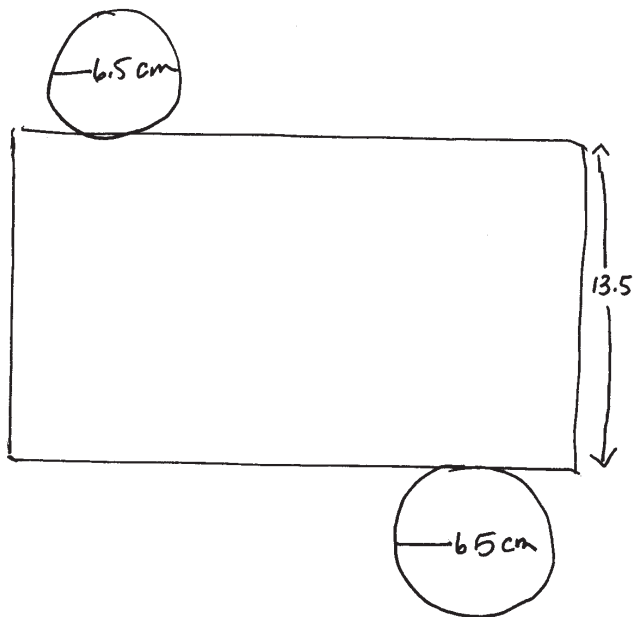
We were given the information that ② the box should be .5 cm bigger at the highest and widest parts. To find these measures we did the following things. The radius of the circle at the top of the bottle is 1.5 cm. The diameter would be 3 cm.

Therefore the height of the bottle is 13 cm. The height of the box must be 13.5 cm (13 + .5). The width is 6 cm on the bottle.



We found that by splitting the hexagon into equilateral triangles the width was 6 cm (3 + 3) to make it as big as needed the width would need to be 6.5 cm on the packing box.

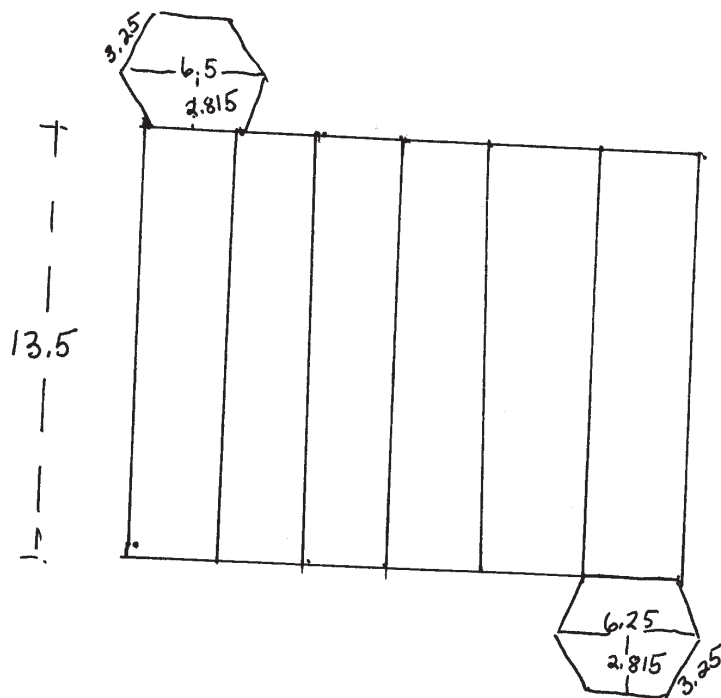
Angles $180(b-2) = \frac{720}{6} = 120$ $\frac{120}{2} = 60^\circ$ each angle.



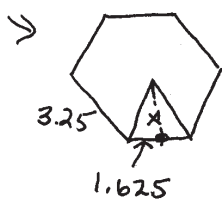
① Formula of Surface Area (SA) of a cylinder is $2\pi rh + \pi r^2$. So we know the height is $h = 13.5$ and the radius is $r = 3.25$ So we fill in the formula accordingly, and solve.

$$\begin{aligned} SA &= 2\pi(3.25)(13.5) + 2\pi(3.25)^2 \\ SA &= 2\pi(43.875) + 2\pi(10.563) \\ SA &= (137.8) + 33.2 \\ SA &= 341.9 \text{ cm}^2 \end{aligned}$$

③



Formula of surface Area (SA) of a hexagonal prism is $SA = Ph + 2B$. So we know the perimeter, P , of the prism is $6(3.25) = 19.5$ and the height $h = 13.5$. In addition, area of a hexagon equals $SA = \frac{1}{2} Pa$ where P stands for perimeter and a stands for apothem. Since $P = 19.5$ and $a = 2.815$



then we use Pythagorean Theorem

$$3.25^2 = 1.625^2 + x^2$$

$$x^2 = 7.421875$$

$$x = \sqrt{7.421875}$$

$$x = 2.815$$

So we fill in the formula accordingly...

$$\text{Area of the hexagon} = \frac{1}{2} Pa$$

$$A = \frac{1}{2} (19.5)(2.815)$$

$$A = 27.4$$

Now we know the area of the hexagon or base we fill in the SA of the prism formula.

$$SA = Ph + 2B$$

$$SA = (19.5)(13.5) + 2(27.4)$$

$$SA = 318.1 \text{ cm}^2$$

The hexagonal prism would serve as the most cost efficient container because it has a lesser surface area than the cylinder.

Playing with Pipes

R&B Engineers have been hired to design a piping system for an industrial plant. The design calls for four different-shaped pipes that will carry water to the industrial plant. The pipes will be constructed out of four rectangular pieces of sheet metal, each with a width of 360 cm and a length of 100 cm. The metal will be folded or rolled to make the four pipes, each having cross sections with the same perimeter, 360 cm. One of the pipes has a rectangular cross section with dimensions of 60 cm by 120 cm. One pipe cross section is a square, one is a regular hexagon, and one is circular. The length of each pipe will be 100 cm.

1. Draw a sketch of each pipe, and label the dimensions on each cross section of pipe.
2. The pipe with the greatest flow is considered to be the pipe with the greatest cross section area. Determine which cross section of pipe will allow for the greatest flow of water, and list them in order from greatest to least. Justify your answers.



Notes

Materials:

One graphing calculator per student

Geometry TEKS Focus:

(G.6) **Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional geometric figures and related two-dimensional representations and uses these representations to solve problems.

The student is expected to:

(C) use orthographic and isometric views of three-dimensional geometric figures to represent and construct three-dimensional geometric figures and solve problems.

(G.8) **Congruence and the geometry of size.** The student uses tools to determine measurements of geometric figures and extends measurement concepts to find perimeter, area, and volume in problem situations.

The student is expected to:

(A) find areas of regular polygons, circles, and composite figures;

Additional Geometry TEKS:

(G.6) **Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional geometric figures and related two-dimensional representations and uses these representations to solve problems.

Teacher Notes

Scaffolding Questions:

- How can the perimeter of the square cross section be determined? The regular hexagonal cross section? The circular cross section?
- How can you determine the amount of water flow through the pipes?

Sample Solutions:

1. All of the pipes have a cross section with the same perimeter. The given perimeter is 360 cm. The dimensions of the rectangular cross section are given in the problem. The dimensions of the other cross sections can be calculated as follows:

$$\text{Square: } \frac{360 \text{ cm}}{4} = 90 \text{ cm per side}$$

$$\text{Regular Hexagon: } \frac{360 \text{ cm}}{6} = 60 \text{ cm per side}$$

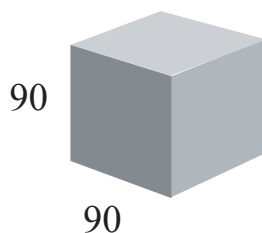
$$\text{Circle: } 2\pi r = 360$$

$$\pi r = 180$$

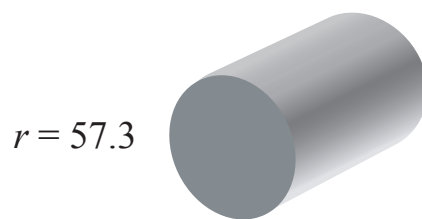
$$\pi r \approx 57.3 \text{ cm}$$

The pipes are shown below.

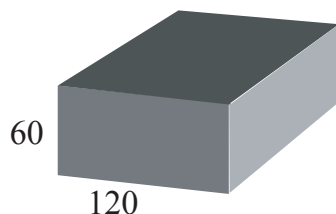
Square



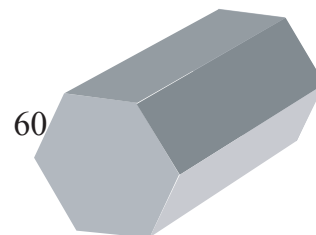
Circular



Rectangle



Regular Hexagon



2. In order to determine the pipe with the greatest water flow, the area of each cross section must be calculated.

Listed in order, from greatest flow to least flow:

The area of the circular region will produce an opening that is: $A = \pi r^2 = \pi(57.3)^2 = 10,314 \text{ cm}^2$.

The area of the regular hexagonal region will produce an opening that is: $A = \frac{1}{2} ap = \frac{1}{2} \cdot 30\sqrt{3} \cdot 360 \approx 9353 \text{ cm}^2$

The area of the square region will produce an opening that is: $A = s^2 = 90 \cdot 90 = 8,100 \text{ square cm}$.

The area of the rectangular region will produce an opening that is: $A = lw = 60 \cdot 120 = 7,200 \text{ square cm}$.

Extension Questions:

- A pipe having a circular cross section with an inside diameter of 10 inches is to carry water from a reservoir to a small city in the desert. Neglecting the friction and turbulence of the water against the inside of the pipes, what is the minimum number of 2-inch inside-diameter pipes of the same length that is needed to carry the same volume of water to the small city in the desert? Explain your answer.

In order to solve this problem, it will be necessary to find the number of smaller pipes that it will take to provide the same cross sectional area as that of the larger pipe.

The cross sectional areas of the pipes are circles having the area $A = \pi r^2$.

The radius of the 10-inch pipe is 5 inches, and the radius of the 2-inch pipe is 1 inch.

The area of the 10-inch pipe is $A_1 = \pi 5^2 = 25\pi \text{ in}^2$

The area of the 2-inch pipe is $A_2 = \pi 1^2 = 1\pi \text{ in}^2$

The ratio of the areas is $25\pi : 1\pi$ or $25 : 1$. This implies that the cross sectional area of the 10-inch pipe is 25 times larger than the cross sectional area of the smaller pipe. It will require 25 of the 2-inch pipes to provide the same cross sectional area as the 10-inch pipe.

The student is expected to:

(A) describe and draw the intersection of a given plane with various three-dimensional geometric figures;

Connections to TAKS:

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

An alternate approach is to realize that the ratio of the diameters is 10 to 2 or 5 to 1. Thus, the ratio of the areas is 5^2 to 1^2 or 25 to 1. Therefore, the requirement is 25 of the 2-inch pipes.

- Suppose the cross sections of the pipes are squares rather than circles. The diagonal of one square is 10 inches, and the diagonal of the second square is 2 inches. Will the shape of the pipes affect the number of pipes required to produce equal areas? Justify your answer.

It makes no difference what shapes the pipes are, provided that the cross sectional areas are the same. If the pipes had square cross sections, and the diagonals were 10 inches and 2 inches, it will still require 25 smaller pipes to provide the same cross sectional area as the larger pipe. The pipe with the 10-inch diagonal forms two 45-45-90 triangles with leg lengths of $5\sqrt{2}$ inches. Therefore, the square cross section will have side lengths of $5\sqrt{2}$ inches. The area of the pipe with the 10-inch diagonal will be $A = (5\sqrt{2})^2 = 50 \text{ in}^2$.

The pipe with the 2-inch diagonal forms two 45-45-90 triangles with leg lengths of $1\sqrt{2}$ inches. Therefore the square cross section will have side lengths of $1\sqrt{2}$ inches. The area of the pipe with the 2-inch diagonal will be $A = (1\sqrt{2})^2 = 2 \text{ in}^2$.

The ratio of the areas of the square cross section is 50 : 2 or 25: 1. Therefore, it will require 25 of the smaller square pipes to provide the same cross sectional area as the larger pipe.

Circular Security

The Circular Security Company manufactures metal cans for small amounts of medical hazardous waste. They have contracted your marketing firm to create a label for the waste can. The can is a right circular cylinder with a base radius of 9 inches and a height of 15 inches.

The slogan for Circular Security is “*Circular* — For that all-around sense of security!” As an employee of the marketing company, your task is to design a label for the can that has a relationship to the slogan. You have decided that a spiral stripe is to be painted on the label of the can, winding around it exactly once as it reaches from bottom to top. It will reach the top exactly above the spot where it left the bottom.

1. Create a net showing the dimensions of the can and the placement of the stripe. Round dimensions to the nearest tenth of an inch if needed.
2. Determine the length (in inches) of the stripe. Round your answer to the nearest tenth of an inch if needed.
3. Justify your solution.



Notes

Materials:

One graphing calculator per student

Geometry TEKS Focus:

(G.6) **Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional geometric figures and related two-dimensional representations and uses these representations to solve problems.

The student is expected to:

(B) use nets to represent and construct three-dimensional geometric figures; and

Additional Geometry TEKS:

(G.6) **Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional geometric figures and related two-dimensional representations and uses these representations to solve problems.

The student is expected to:

(C) use orthographic and isometric views of three-dimensional geometric figures to represent and construct three-dimensional geometric figures and solve problems.

Teacher Notes

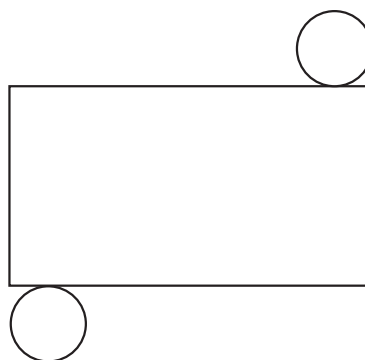
Scaffolding Questions:

- What two-dimensional figures comprise a net of a cylinder?
- If the radius of the cylinder is 9 inches, what is the circumference of the cylinder?
- How will knowing the circumference of the cylinder help you determine the dimensions of the net?
- Where would the stripe be on the net of the cylinder?
- How can you calculate the length of the stripe?

Sample Solutions:

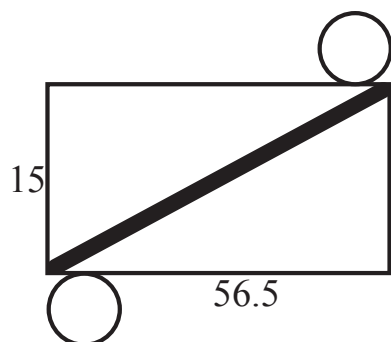
This performance task begins as a three-dimensional scenario but can be reduced to a two-dimensional scenario to make the solution easier. Some cans, like ones that might be found in a grocery store, have a label all the way around them. If the label is removed and flattened out, the shape of the label is a rectangle. The top and the bottom of the can are circles.

The net for the can is:



The height of the can is 15 inches; this will also be the height of the rectangle representing the label. The radius of the circular base of the can is used to find the circumference of the circle, which will turn out to be the width of the rectangular label.

If $r = 9$ inches, then the circumference of the circle is $2\pi(9) \approx 56.5$ inches. Therefore the label's width is also 56.5 inches.



Let the length of the diagonal be represented by x . The diagonal black line in the net above represents the stripe. The problem states that the stripe goes from the top to the bottom of the can in exactly one circumference; on the rectangle this is the length of the diagonal.

The Pythagorean Theorem can be used to find the length of the diagonal as follows:

$$\begin{aligned} 15^2 + 56.5^2 &= x^2 \\ 225 + 3192.25 &= x^2 \\ 58.5 &\approx x \end{aligned}$$

The length of the diagonal stripe, from corner to corner, is approximately 58.5 inches.

Extension Questions:

- Suppose a Circular Security customer wants to place a special order for a can that is similar to the original can but will have a height of 10 inches. Determine the dimensions of the new can and the length of the stripe that will encircle the can.

If a customer wants to order a similar can with a height of 10 inches, the ratio of the corresponding sides is 15 to 10, or 3 to 2. To find the radius of the base, solve the proportion.

$$\begin{aligned} \frac{3}{2} &= \frac{9}{x} \\ x &= 6 \text{ inches.} \end{aligned}$$

The radius of the special-order cylinder is 6 inches. The circumference of the base is the same as the width of the rectangle.

(G.8) **Congruence and the geometry of size.** The student uses tools to determine measurements of geometric figures and extends measurement concepts to find perimeter, area, and volume in problem situations.

The student is expected to:

(C) derive, extend, and use the Pythagorean Theorem; and

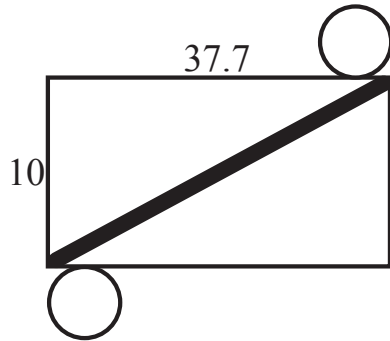
Connections to TAKS:

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

$$C = 2\pi(6)$$

$$C \approx 37.7 \text{ inches}$$



The length of the stripe encircling the special order can is represented by x :

$$10^2 + 37.7^2 = x^2$$

$$100 + 1421.29 = x^2$$

$$1521.29 = x^2$$

$$x \approx 39.0 \text{ inches}$$

- Compare the stripe of the original can to this new can's stripe-length.

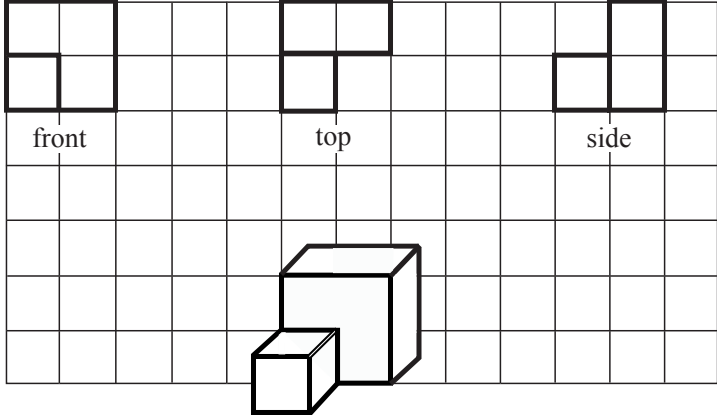
The ratio of the original stripe-length to the new stripe-length is 58.5 in to 39.0 in.

$$\frac{58.5}{39.0} = \frac{1.5}{1} = \frac{3}{2}$$

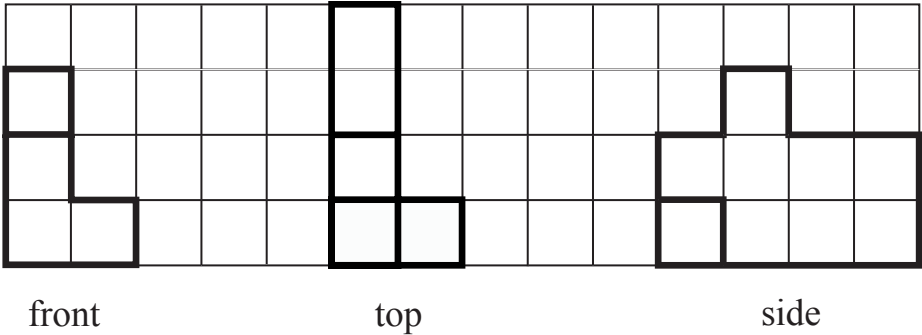
The ratio of the stripes is the same as the ratio that compares the dimensions of the two cans or 3 to 2.

Different Views

1. The picture below shows the front, top, and side views of an object as well as the object itself. This object is made of five unit cubes (four in the back row and one in the front), so the volume is 5 units³. The surface area is 20 units².



2. Below are the front, top, and side views of another object. Draw this object, and explain how the given views helped you construct your final object. Calculate the volume of this object, and carefully explain how you got your answer.





Notes

Materials:

Cubes or blocks for each student (optional)

Geometry TEKS Focus:

(G.6) Dimensionality and the geometry of location. The student analyzes the relationship between three-dimensional geometric figures and related two-dimensional representations and uses these representations to solve problems.

The student is expected to:

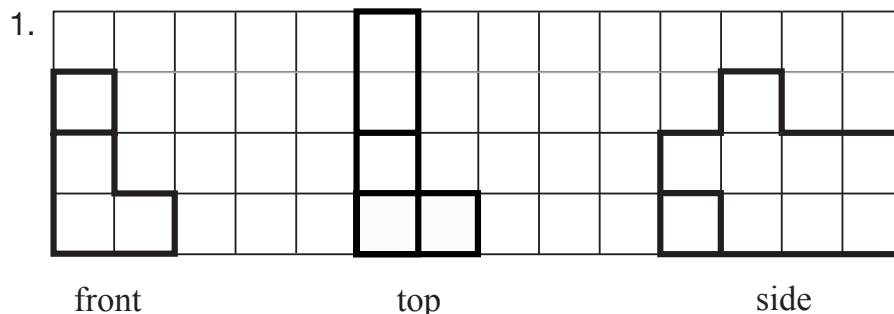
(C) use orthographic and isometric views of three-dimensional geometric figures to represent and construct three-dimensional geometric figures and solve problems.

Teacher Notes

Scaffolding Questions:

- What does the square on the front view of the very first figure tell you?
- Demonstrate your answer above by using cubes.
- Build the final figure in the first set of diagrams.
- How many cubes make up the final figure?
- What is the volume of one of the cubes?
- How do you know that the volume of the first final figure really is 5 cubic units?
- Determine the surface area of one cube.
- Explain how you know the surface area of the first final figure really is 20 square units.

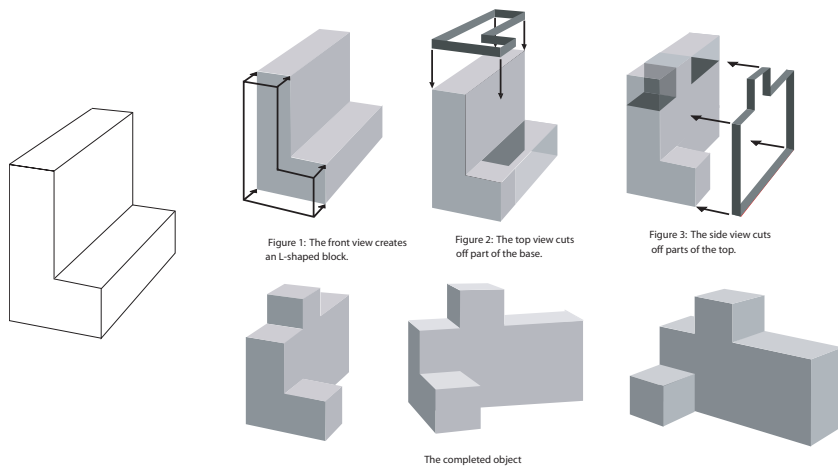
Sample Solutions:



The front view of the object has three blocks in an L shape and another block on top that is either farther in or farther out.

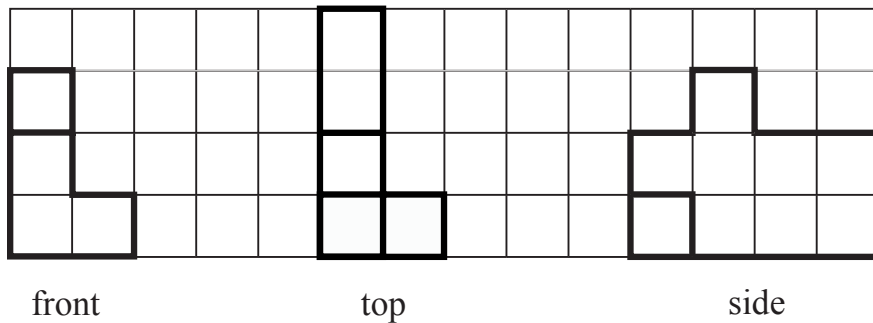
The top view indicates that the front three squares have different heights and that there are blocks at the end that are of the same height. The top view also produces an overall L shape.

The side view indicates that there is one block that sticks out. This is also visible from the front and top views. In the side view there is also a wall two blocks high and four blocks long. There is a tower on top of the wall consisting of 1 block. From this information, a picture of the figure is drawn.

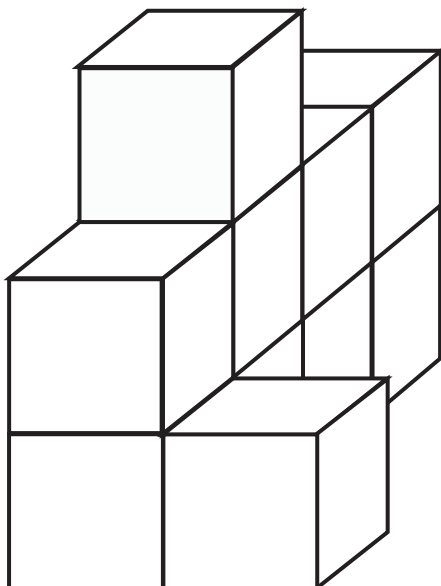


2. Using the given information in the original diagrams, a sketch is drawn showing the individual blocks.

Given Diagram:



New figure with individual blocks drawn:



Additional Geometry TEKS:

(G.6) **Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional geometric figures and related two-dimensional representations and uses these representations to solve problems.

The student is expected to:
(B) use nets to represent and construct three-dimensional geometric figures; and

Connections to TAKS:

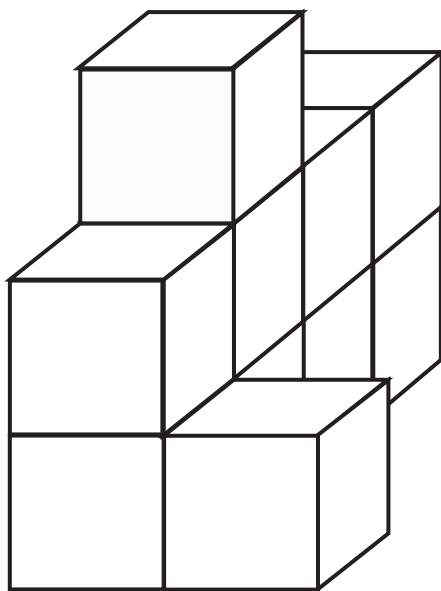
Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Using the new figure, the total number of blocks is determined as follows. Consider each row to be a tower if the blocks are separated. There are three blocks in the front row (the L-shaped front view), three blocks are in the second row (a three-block tower), two blocks in the third row, and two blocks in the fourth row. The total number of blocks is $3 + 3 + 2 + 2 = 10$ blocks. Each block is 1 cubic unit; therefore, the figure has a volume of 10 cubic units.

Extension Question:

- Find the surface area of the second object. Justify your answer.

To find the surface area, the figure is divided into rows, front to back.



Each face on a cube that is showing adds a square unit to the surface area of the figure. Faces that touch each other are not considered part of the surface area. The bottom, back, and left side of the figure is part of the surface area, even though they are not showing.

There are 3 blocks in the front row (the L-shaped front view.) This tower has 3 faces showing on the front, 1 face showing on the back, 2 on the left side, 2 on the right side, and 2 on top. There are 2 faces on the bottom of the tower. There are a total of 12 faces on the front of the tower.

The second row is a 3-block tower. This tower has 1 face showing from the front, 1 face on the back, 1 face on the bottom, 1 face on the top, 3 faces on the left side, and

3 faces on the right side. There are a total of 10 faces on the second tower.

The third row is a tower, 2 blocks high. There are no faces showing from the front or the back. There is 1 face on the top, 1 face on the bottom, 2 faces on the left side, and 2 faces on the right side. There are a total of 6 faces on the third tower.

The fourth row is a tower that is also 2 blocks high. There are no faces showing from the front. There are 2 faces on the back, 1 face on the bottom, 1 face on the top, 2 faces on the left, and 2 faces on the right. There are a total of 8 faces on the fourth row.

The total number of faces that composes the surface area is found by adding the total from each row: $12 + 10 + 6 + 8 = 36$ faces. The surface area of this figure is 36 square units.

In alternate approach, the surface area can be found by subtracting the number of faces that are facing each other from the total number of faces of the 10 cubes.

The total number of faces of 10 cubes is $10 \cdot 6$ because each cube has 6 faces.

There are 12 cube faces that are touching another face. The cube at the front right faces another cube. When row two is stacked on row one, there are 4 horizontal faces touching each other and 6 vertical faces touching each other. There is one more face touching another face when row three is stacked. Thus, the total number is $1 + 4 + 6 + 1$ or 12 faces.

The total number of facing cubes is $12 \cdot 2$ or 24 faces.

The number of faces in the surface area is $60 - 24$ or 36 faces. Therefore, the surface area of the figure is 36 square units.

The Slice Is Right!

A cube has edges that are 4 units in length. At each vertex of the cube, a plane slices off the corner of the cube and hits each of the three edges at a point 1 unit from the vertex of the cube.

1. Draw the cube showing how the corners will be sliced.
2. Draw and label a figure that represents the solid that is cut from each corner of the cube.
3. How many cubic units are in the volume of the solid that remains? Justify your answer.



Notes

Materials:

One graphing calculator per student

Styrofoam or clay cubes, fishing line to cut off corners

Geometry TEKS Focus:

(G.6) **Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional geometric figures and related two-dimensional representations and uses these representations to solve problems.

The student is expected to:

(A) describe and draw the intersection of a given plane with various three-dimensional geometric figures;

Additional Geometry TEKS:

(G.8) **Congruence and the geometry of size.** The student uses tools to determine measurements of geometric figures and extends measurement concepts to find perimeter, area, and volume in problem situations.

The student is expected to:

(D) find surface areas and volumes of prisms, pyramids, spheres, cones, cylinders, and composites of these figures in problem situations.

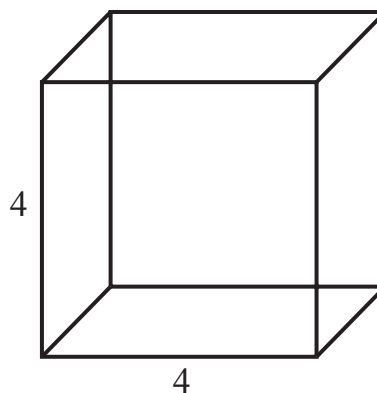
Teacher Notes:

Scaffolding Questions:

- If the plane cuts off the corner one unit from the edge of the cube, what is the shape of the piece that is cut off?
- How many of the corners are cut off?
- What are the dimensions of each piece that is cut off?
- Determine the volume of the original cube.
- Determine the volume of each cut off piece.

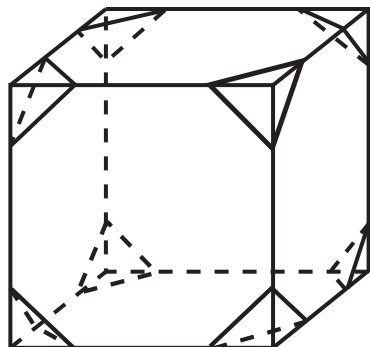
Sample Solutions:

1. Original Cube:

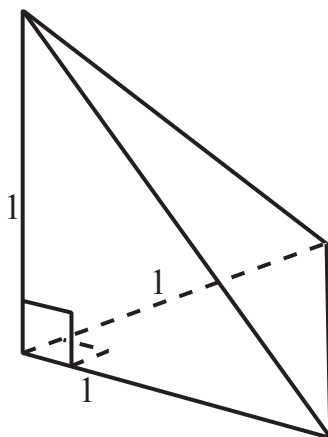


When the corners of the cube are cut off, eight congruent pyramids with four triangular faces are formed. Three of the faces are isosceles right triangles with legs that are 1 unit in length. The fourth face is an equilateral triangle. The side of this triangle is also the hypotenuse of the isosceles right triangle.

A cube with all corners sliced by a plane at a distance of 1 unit from each corner:



2. Each triangular pyramid may be positioned so that its base is one of the isosceles right triangles. The height of the pyramid is also 1 unit. The corner of the cube forms a right angle on each face of the pyramid.



3. Each triangular pyramid that is sliced off has volume:

$$V = \frac{1}{3} Bh = \frac{1}{3} \left(\frac{1}{2} \cdot 1 \cdot 1 \right) \cdot 1 = \frac{1}{6}$$

The volume of the eight pyramids that are sliced off the cube is:

$$8 \cdot \frac{1}{6} = \frac{8}{6} = \frac{4}{3}$$

The volume of the original cube is: $V = 4^3 = 64$ cubic units.

The volume of the solid after the eight corners are removed can be found by subtracting the volume of the

Connections to TAKS:

Objective 7: The student will demonstrate an understanding of two-and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

One teacher says . . .

“Some students who don’t do as well on traditional assignments were in their element with this problem. It pleased me that several students saw how you would turn the corner pieces around in order to find the volume before I asked any scaffolding questions. It worked well to have them put their solutions on chart paper. They got a chance to be thorough with their thinking as well as their creativity.”

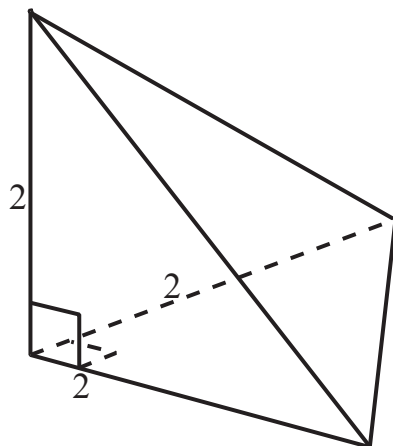
eight pyramids from the volume of the original cube.

$V = \text{original cube} - 8 \text{ pyramid volumes}$

$$V = 64 - \frac{4}{3} = 62\frac{2}{3} \text{ cubic units}$$

Extension Question:

- Suppose the plane slices off sections that are 2 units from each vertex of the cube. How does the volume of the remaining figure compare to the original slice (1 unit from each vertex)? Calculate the percent of decrease of the volume. Round your answer to the nearest percent. Justify your answer.



There are eight corners on the original cube; therefore, eight triangular pyramids are formed. Each triangle pyramid has an isosceles right triangle for the base and two faces. The area of the base triangle is $\frac{1}{2} \cdot 2 \cdot 2$ or 2 square units.

The volume of one pyramid is $\frac{1}{3} \cdot 2 \cdot 2$ or $\frac{4}{3}$ cubic units.

The volume of the eight pyramids is $8\left(\frac{4}{3}\right)$ or $\frac{32}{3}$ or $10\frac{2}{3}$ cubic units.

The volume of the solid after the eight unit corners have been removed is $64 - 10\frac{2}{3}$ or $53\frac{1}{3}$.

The difference of the volume of the solid after the eight 2-unit corners are removed and the cube with the eight 1-unit pyramids removed can be found by subtracting the volumes.

$V =$ volume of cube with eight 1-unit pyramids – volume of cube with eight 2-unit pyramids

$$V = 62\frac{2}{3} - 53\frac{1}{3} = 9\frac{1}{3} \text{ cubic units}$$

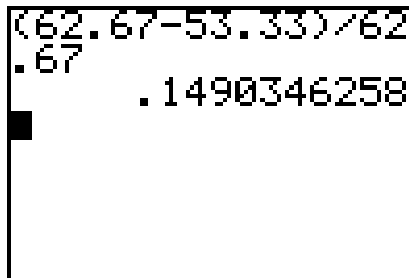
The percent of decrease in the volume of the cubes is found by calculating the following:

$$\frac{(\text{original volume}) - (\text{new volume})}{(\text{original volume})} \cdot 100$$

Fractions were converted to decimals for ease of use in calculation:

Original volume with 1-inch slices: $62\frac{2}{3} = 62.67$ cubic units

New volume with 2-inch slices: $53\frac{1}{3} = 53.33$ cubic units



The percent of volume decrease is rounded to 15%.

Student Work Sample

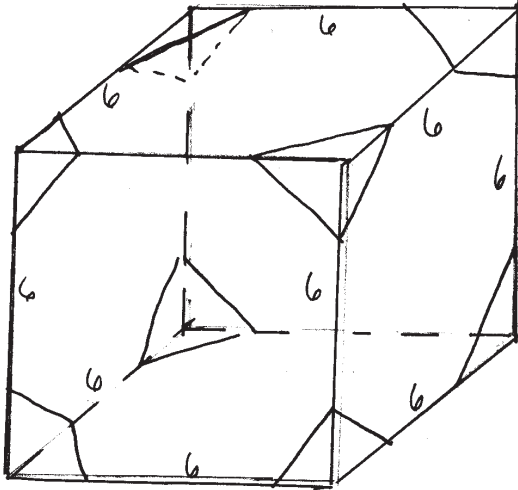
The student who created the work on the next page

- Shows an understanding of the relationships among elements.

The student understands what the given information means and how to interpret the given to create the solid that is cut off from the corners of the cube. He correctly draws the resulting triangular pyramid.

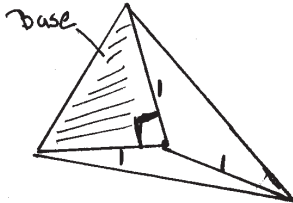
- Demonstrates geometric concepts, processes, and skills.

The student shows the formulas used to determine area and volume. He shows the triangular pyramid, shows the base of the pyramid, and how to determine the volume using the right triangle base.



A cube has edges of length 6 units at each vertex of the cube a plane slices off the corner of the cube and hits each of three edges at a point 1 unit from the vertex of the cube.

The volume of the original cube is: $6^3 = 216 \text{ units}^3$



Area of one of the cut off corners.
+
Volume

Area of Base: $\frac{\text{Base} \cdot \text{Height}}{2} = \frac{1 \cdot 1}{2} = 0.5 \text{ units}^2$

Volume: $0.5 \cdot 1 = \frac{0.5}{3} = \frac{1}{6} \text{ units}^3$

The Volume of one of these pyramids is $\frac{1}{6} \text{ units}^3$

So the added volume of all these pyramids combined is $1 \cdot \frac{1}{3} \text{ units}^3$

- Subtract the volume of the corners from the volume of the cube

$$\begin{array}{r} 216 \\ - 1\frac{1}{3} \\ \hline 214\frac{2}{3} \text{ cubic units} \end{array}$$

Chapter 6:
Congruence



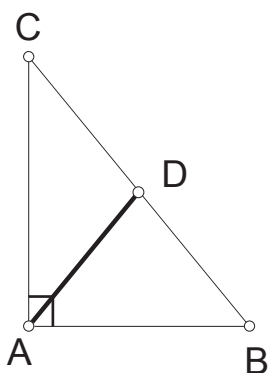
Introduction

Application assessments can provide the context for using geometry and for applying problem-solving techniques. The six tasks in this chapter make connections among geometric concepts. The students will use the concept of congruence to solve problems. For example, The Shortest Cable Line involves a reflection of a triangle. Tell Me Everything You Can About... requires constructions. The School Flag assesses area concepts. Median to the Hypotenuse of a Right Triangle requires conjecturing and justification using axiomatic or coordinate methods.



Median to the Hypotenuse of a Right Triangle

1. On a right triangle, how does the length of the median drawn to the hypotenuse compare with the length of the hypotenuse? Use at least two approaches—drawings, constructions, or appropriate geometry technology—to investigate and conjecture a relationship between these lengths.



Triangle ABC is a right triangle with $\overline{AC} \perp \overline{AB}$. \overline{AD} is the median to the hypotenuse.

2. Prove your conjecture using one of these methods:
 - a) congruence transformations (isometries)
 - b) a Euclidean argument
 - c) a coordinate proof
3. Is your conjecture for right triangles valid for other triangles? Why or why not?
4. Is the converse of your conjecture true? Why or why not?



Notes

Materials:

One compass and ruler per student

Unlined paper, patty paper, and geometry software

Geometry TEKS Focus:

(G.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student is expected to:

(B) make conjectures about angles, lines, polygons, circles, and three-dimensional figures and determine the validity of the conjectures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(G.3) **Geometric structure.** The student applies logical reasoning to justify and prove mathematical statements.

The student is expected to:

(B) construct and justify statements about geometric figures and their properties;

Teacher Notes

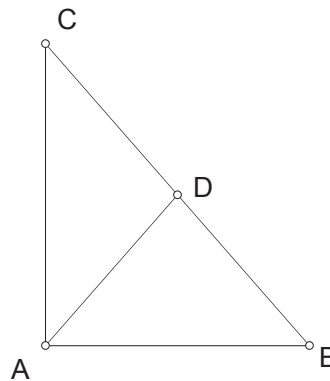
Scaffolding Questions:

- What is meant by a median of a triangle?
- Construct a scalene right triangle, an isosceles right triangle, and a 30-60-90 triangle. Construct the median to the hypotenuse. What appears to be true about these medians? How can you use measurements to investigate your observation?
- What is meant by congruence transformations? What are these transformations?
- What is meant by a Euclidean argument?
- A right triangle can always be viewed as “half” of what polygon? How could you extend (add auxiliary segments) the diagram of the right triangle to show this?

Sample Solutions:

Use geometry technology or paper and pencil constructions and take measurements of the median to the hypotenuse and the hypotenuse. For example,

$AD = 3.54$ cm
 $CD = 3.54$ cm
 $DB = 3.54$ cm
 $CB = 7.08$ cm



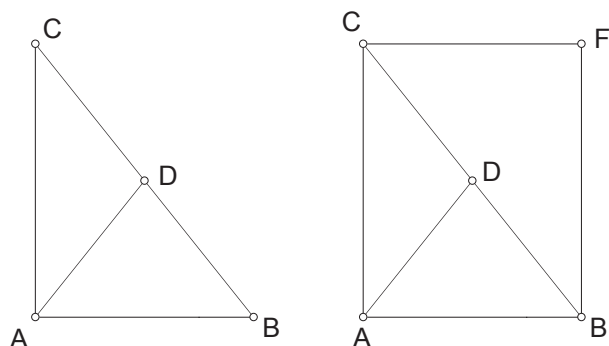
In $\triangle BAC$, $\overline{BA} \perp \overline{AC}$ and \overline{AD} is the median to the hypotenuse.

1. Conjecture: The measurements show that the median to the hypotenuse is one-half as long as the hypotenuse, or that the hypotenuse is twice as long as the median

drawn to it.

2. Congruence Transformations:

Rotate $\angle ACB$ 180 degrees clockwise about point D and $\angle ABC$ 180 degrees counter-clockwise about point D. Label the point of intersection of the angles point F.



These transformations form $\angle FBC$ and $\angle FCB$ so that $\angle ACB \cong \angle FBC$ and $\angle ABC \cong \angle FCB$

This means that \overline{AC} is parallel to \overline{BF} and \overline{AB} is parallel to \overline{CF} because these angle pairs are congruent alternate interior angles.

Additionally, this means that quadrilateral ABFC is a parallelogram. Since $\angle CAB$ is a right angle, we know that all four angles are right angles.

Now we know that ABFC is a rectangle. Since the diagonals of a parallelogram bisect each other, and the diagonals of a rectangle are congruent, we now know that

$$AD = \frac{1}{2}AF = \frac{1}{2}BC$$

This proves our conjecture that on a right triangle the median drawn to the hypotenuse is half as long as the hypotenuse.

Additional Geometry TEKS:

(G.1) **Geometric structure.** The student understands the structure of, and relationships within, an axiomatic system.

The student is expected to:

(A) develop an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems;

(G.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student is expected to:

(A) use constructions to explore attributes of geometric figures and to make conjectures about geometric relationships; and

(G.3) **Geometric structure.** The student applies logical reasoning to justify and prove mathematical statements.

The student is expected to:

(A) determine the validity of a conditional statement, its converse, inverse, and contrapositive;

(C) uses logical reasoning to prove statements are true and find counter examples to disprove statements that are false;

(G.10) **Congruence and the geometry of size.** The student applies the concept of congruence to justify properties of figures and solve problems.

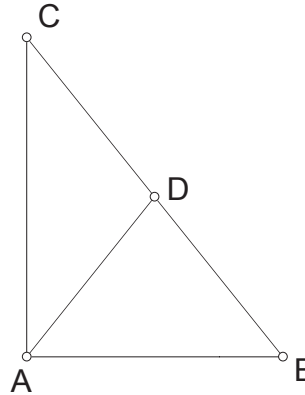
The student is expected to:

(A) use congruence transformations to make conjectures and justify properties of geometric figures including figures represented on a coordinate plane; and

Connections to TAKS:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

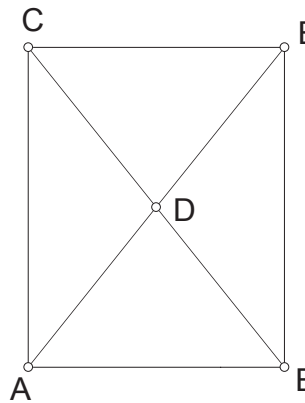
Euclidean Argument:



We are given that $\triangle BAC$ is a right triangle with a right angle at A and that \overline{AD} is the median to the hypotenuse.

By the definition of a median, we know that $CD = DB$.

Extend \overline{AD} through point D to point E so that $AD = DE$, and draw \overline{EC} and \overline{EB} to form quadrilateral $ACEB$.



By the vertical angle theorem, $\angle ADC \cong \angle EDB$.

Now $\triangle ADC \cong \triangle EDB$ by SAS.

It follows that $\overline{AC} \cong \overline{EB}$ and $\angle CAD \cong \angle BED$ because these are corresponding parts of congruent triangles.

These congruent angles are alternate interior angles formed by transversal \overleftrightarrow{AE} on \overleftrightarrow{AC} and \overleftrightarrow{EB} . Therefore, \overleftrightarrow{AC} and \overleftrightarrow{EB} are parallel.

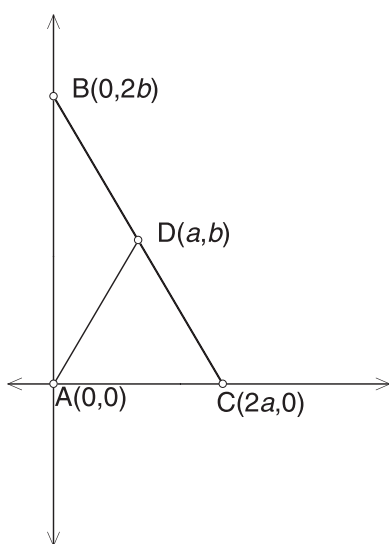
Now consider quadrilateral $ACEB$. Since quadrilateral

ACEB has a pair of opposite sides that are congruent and parallel, the quadrilateral is a parallelogram.

Next, since opposite angles of a parallelogram are congruent, and consecutive angles are supplementary, and since $\angle BAC$ is a right angle, the other angles of the parallelogram are right angles. This makes ACEB a rectangle.

Finally, the diagonals of a rectangle bisect each other and are congruent so that $AD = \frac{1}{2} AE = \frac{1}{2} BC$.

Coordinate Proof:



Draw right triangle BAC with right angle A at the origin and the legs along the axes.

Assign to vertex C the coordinates $(2a, 0)$ and to vertex B the coordinates $(0, 2b)$.

We know that \overline{AD} is the median to the hypotenuse, which means point D is the midpoint of \overline{BC} . We apply the midpoint formula to get the coordinates of point D.

$$\left(\frac{2a+0}{2}, \frac{0+2b}{2} \right) = (a, b)$$

Notes

Now we use the distance formula to find the lengths of the median and the hypotenuse.

$$AD = \sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2 + b^2}$$

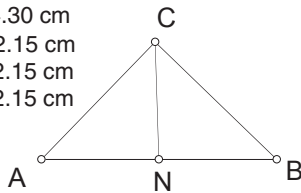
and

$$\begin{aligned} BC &= \sqrt{(2a-0)^2 + (0-2b)^2} \\ &= \sqrt{4a^2 + 4b^2} \\ &= \sqrt{4(a^2 + b^2)} \\ &= 2\sqrt{a^2 + b^2} \end{aligned}$$

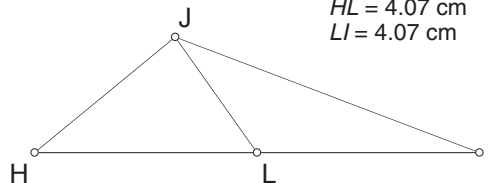
This shows that $AD = \frac{1}{2}BC$; that is, on a right triangle the median drawn to the hypotenuse is half as long as the hypotenuse.

3. Other triangles to consider are an acute triangle and an obtuse triangle. Using geometry technology or paper and pencil constructions, we can easily investigate this situation. Possible values are given below.

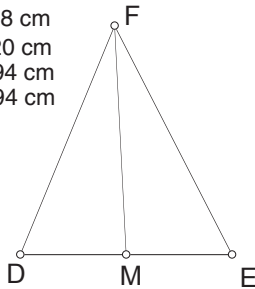
$m\angle ACB = 90^\circ$
 $AB = 4.30$ cm
 $CN = 2.15$ cm
 $AN = 2.15$ cm
 $NB = 2.15$ cm



$m\angle HJI = 120^\circ$
 $HI = 8.15$ cm
 $JL = 2.59$ cm
 $HL = 4.07$ cm
 $LI = 4.07$ cm



$m\angle DFE = 50^\circ$
 $DE = 3.88$ cm
 $FM = 4.20$ cm
 $DM = 1.94$ cm
 $ME = 1.94$ cm



$\triangle ABC$ is a right triangle and demonstrates the conjecture we made and proved.

$\triangle DFE$ shows the median drawn from an acute angle to the opposite side and shows that the median is longer than half of the side to which it is drawn.

$\triangle HJI$ shows the median drawn from an obtuse angle to the opposite side. It shows that the median is shorter than half of the side to which it is drawn, but it is not half the length of that side.

This suggests the following conjecture:

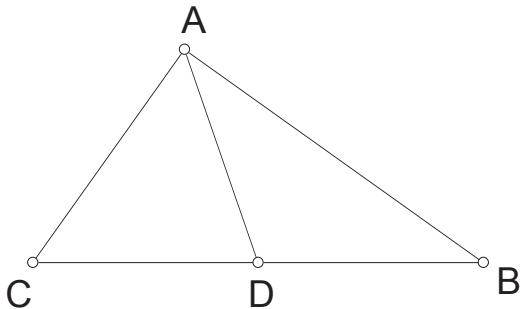
If a median on a triangle is drawn from an acute angle, the median will be longer than half of the side to which it is drawn. If a median on a triangle is drawn from an obtuse angle, the length of the median will be less than half the length of the side to which it is drawn.

4. The converse of our conjecture in sample solution 1 is the following:

If a median drawn on a triangle is half as long as the side to which it is drawn, then the median is drawn to the hypotenuse of a right triangle.

This is true, as the following argument shows:

Let \overline{AD} be a median on $\triangle CAB$ and $AD = \frac{1}{2}CB$.



Then, by the definition of a median, D is the midpoint of \overline{CB} , and we have $CD = DB = AD$. This makes $\triangle CDA$ and $\triangle BDA$ isosceles triangles with bases \overline{CA} and \overline{BA} . Since base angles of an isosceles triangle are congruent,

$$m\angle DCA = m\angle DAC = x, \text{ and}$$

$$m\angle DBA = m\angle DAB = y$$

Now the angles of $\triangle CAB$ sum to 180° so that

$$m\angle DCA + m\angle CAB + m\angle DBA = x + (x + y) + y = 180^\circ$$

This simplifies to

$$2x + 2y = 180$$

$$x + y = 90$$

This shows that $\angle CAB$ is a right angle, and $\triangle CAB$ is a right triangle with hypotenuse \overline{CB} .

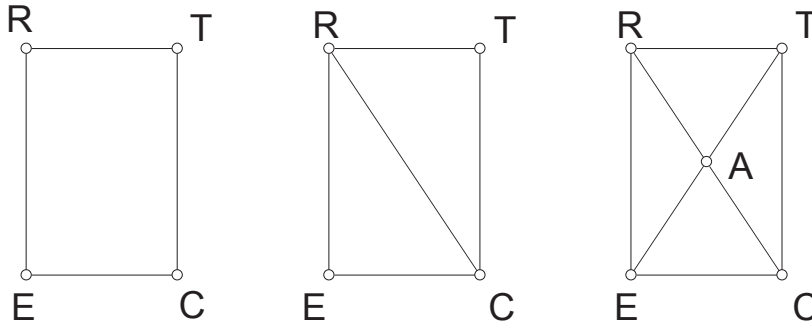
Extension Questions:

- How does this task relate to your work with parallelograms?

A right triangle is formed when a diagonal is drawn on a rectangle. The diagonals of a rectangle are congruent and bisect each other. Therefore, on a right triangle, the median of a right triangle drawn to the hypotenuse is half the diagonal drawn on a rectangle.

Consider rectangle $RECT$. Draw diagonal \overline{RC} to form a right triangle, $\triangle REC$ with hypotenuse \overline{RC} . Draw diagonal \overline{ET} , and label the point of intersection A . Because of the properties of rectangles, $RA = AC = EA = AT$.

Therefore, \overline{EA} becomes the median drawn to the hypotenuse \overline{RC} of $\triangle REC$ and is half as long as \overline{RC} .



- You have shown that on a right triangle the median drawn to the hypotenuse is half as long as the hypotenuse. You were given the choice of three types of proofs. What might be an advantage or disadvantage for each type?

We can write the proof using congruence transformations, a Euclidean argument, or a coordinate proof.

The transformation approach is nice because we can model it with patty paper, and that makes it more visual.

The Euclidean approach is a little harder. We have to start with the right triangle and the median drawn on it and realize that we must draw auxiliary segments on it so that we form a rectangle with its diagonals.

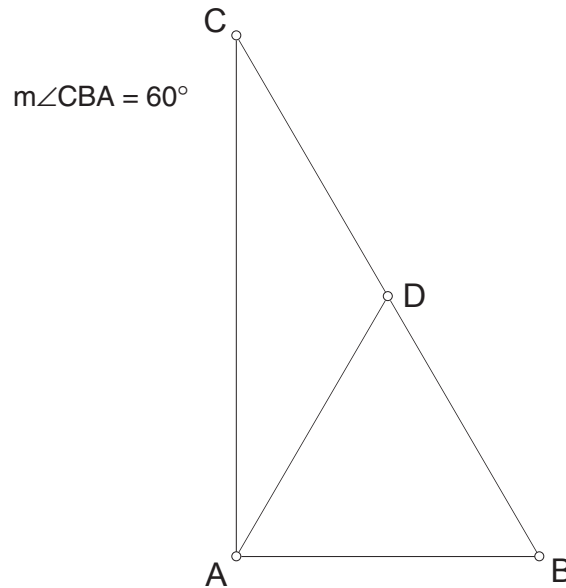
A coordinate proof is nice because we see how we can use algebra to prove geometric concepts.

- Suppose that $\triangle BAC$ is a right triangle with $\overline{BA} \perp \overline{AC}$ and \overline{AD} as the median to the hypotenuse. What do we know about BD , AD , and CD ? How does this relate to your knowledge of circles?

Since $BD = AD = CD$ and these four points are coplanar, we know that points B, A, and C lie on a circle with center D. Since points B, D, and C are collinear, we know that \overline{BC} is a diameter of the circle.

- What additional patterns emerge if we draw the median to the hypotenuse of the two special right triangles (the triangle with angles that measure 30° , 60° , and 90° or 45° , 45° , and 90°)?

Consider a 30-60-90 triangle:



Triangle BAC is a 30-60-90 triangle with $m\angle BAC = 90^\circ$ and $m\angle B = 60^\circ$. \overline{AD} is the median to the hypotenuse.

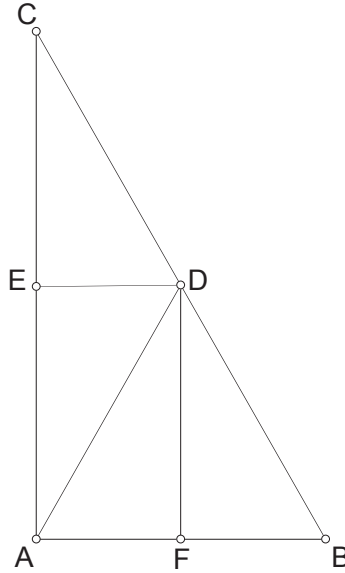
Suppose $BC = 10$. Then $AD = DB = 5$. Also, $AB = 5$. This makes $\triangle ADB$ equilateral.

The other triangle formed, $\triangle ADC$, is an isosceles triangle. The legs are $AD = DC = 5$ and base angles are 30 degrees.

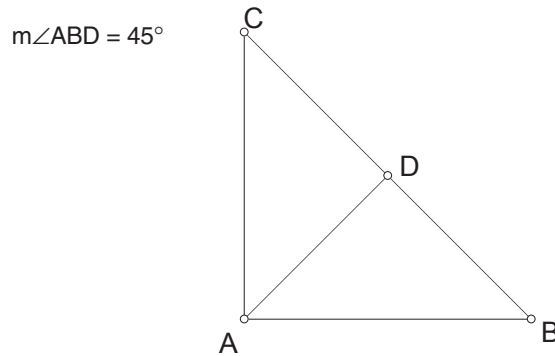
If we draw the median from point D to segment \overline{AB} , this will form two 30-60-90 triangles. Also, if we draw the median from D to \overline{AC} , we will have 30-60-90 triangles.

This second set of medians separates the original triangle ABC into four congruent 30-60-90 triangles.

$BC = 10.11$ cm
 $DA = 5.06$ cm
 $DC = 5.06$ cm
 $BD = 5.06$ cm
 $DE = 2.50$ cm
 $BF = 2.50$ cm
 $FA = 2.50$ cm



Now consider the isosceles right triangle:

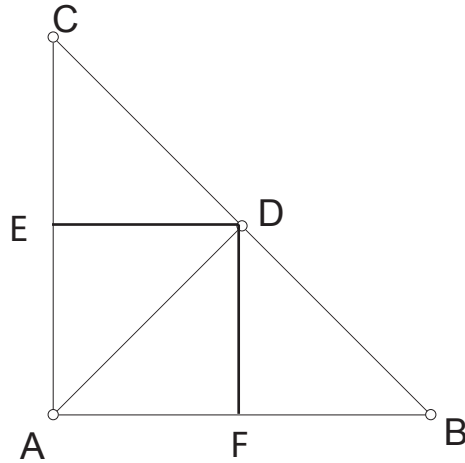


The median to the hypotenuse separates the triangle into two congruent isosceles right triangles:

1. They are isosceles because $AD = DC = DB$.
2. They are right triangles because the median from the vertex angle of an isosceles triangle is also an altitude so that $\overline{AD} \perp \overline{BC}$.
3. $\triangle ADC \cong \triangle ADB$ by SSS because of (1) and because \overline{AC} and \overline{AB} are congruent legs of the original isosceles right triangle.

If we draw the medians from point D to segments \overline{AB} and \overline{AC} , it will separate the original isosceles right triangle into four congruent isosceles right triangles.

$$m\angle ABD = 45^\circ$$

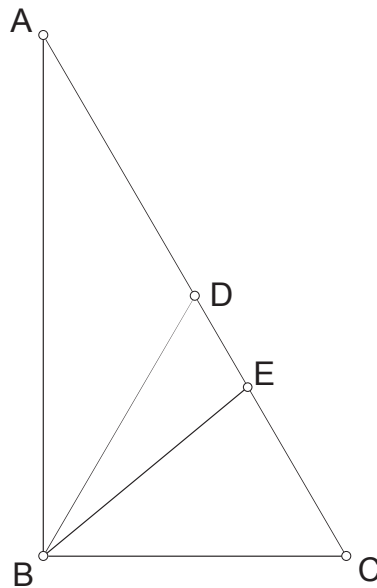


$$\triangle CED \cong \triangle AED \cong \triangle AFD \cong \triangle BFD \text{ by SSS}$$

The isosceles right triangle pattern is different from the 30-60-90 triangle pattern. Drawing the first median to a hypotenuse gives two isosceles right triangles. Drawing the medians on these repeats this pattern.

In the 30-60-90 triangle pattern, the first median drawn to the hypotenuse gives an isosceles triangle (30-30-120) and an equilateral triangle. Drawing medians to the base of isosceles triangle and the equilateral triangle gives four 30-60-90 triangles.

- The median divides the given right triangle into two triangles. Make and prove a conjecture about the areas of the triangles.



The two triangles have equal area. To demonstrate this, drop a perpendicular from B to \overline{AC} to form a segment \overline{BE} . This segment is an altitude to both triangles, with bases \overline{AD} and \overline{DC} . These two segments are equal in length. The triangles have equal bases and the same height, thus they have the same area.

Tell Me Everything You Can About...

1. Given a line segment, \overline{RO} , that is a diagonal of a rhombus RHOM, construct the rhombus. Describe your construction procedure, and explain why your figure is a rhombus.
2. List as many conclusions that you can make about the triangles that result from the construction as possible.

Justify your conclusions.

3. Explain why and how the diagonals may be used to determine the area of the rhombus.



Notes

Materials:

One compass and straightedge per student

Unlined paper, patty paper, and construction tools (including geometry software)

Geometry TEKS Focus:

(G.9) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student is expected to:

(B) formulate and test conjectures about the properties and attributes of polygons and their component parts based on explorations and concrete models;

(G.10) **Congruence and the geometry of size.** The student applies the concept of congruence to justify properties of figures and solve problems.

The student is expected to:

(B) justify and apply triangle congruence relationships.

Additional Geometry TEKS:

(G.1) **Geometric structure.** The student understands the structure of, and relationships within, an axiomatic system.

The student is expected to:

(A) develop an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems;

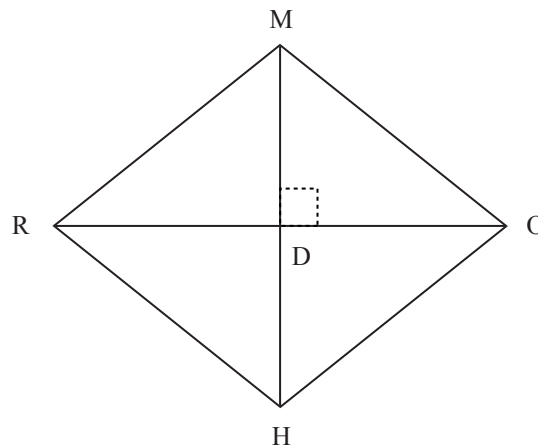
Teacher Notes

Scaffolding Questions:

- What is the definition of a rhombus?
- How is a square related to a rhombus?
- How are the diagonals of a square related?
- How are the diagonals of a rhombus related?
- What special triangles do you see on your constructed rhombus?
- What kind of triangle congruencies do you see?

Sample Solutions:

1.



Construction process using patty paper:

Step 1: To construct the perpendicular bisector of \overline{RO} , fold the patty paper so that R reflects onto O. The fold line is the perpendicular bisector.

Step 2: Mark a point, M, on this perpendicular bisector. A point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment, so $RM = MO$.

Step 3: Locate point H on the perpendicular bisector of \overline{RO} , on the opposite side from point M, so that $RH = RM$. Then, since H is equidistant from R and O, $HO = RH$.

Because $HD = DM$, and \overline{RO} is perpendicular to \overline{HM} , \overline{RO}

is the perpendicular bisector of \overline{HM} . Any point on \overline{RO} is equidistant from H and M. Thus, $HO = MO$. Hence, $HO = MO = RM = RH$. The figure is a rhombus because it is a quadrilateral with four congruent sides.

2. Possible answers:

Triangles RMO, RHO, HRM, and HOM are isosceles triangles because $HO = MO = RM = RH$.

Triangles RMO and RHO are congruent by SAS.

Triangles HRM and HOM are congruent by SAS because $HO = MO = RM = RH$.

Since a rhombus is a parallelogram, its opposite angles are congruent.

Therefore, $\angle RMO \cong \angle RHO$, and $\angle HRM \cong \angle HOM$.

$\triangle RMO \cong \triangle RHO \cong \triangle HRM \cong \triangle HOM$

Since the diagonals of the rhombus are perpendicular, they divide the rhombus into four right triangles with $\angle RMO \cong \angle RHO$ and $\angle HRM \cong \angle HOM$. \overline{HM} and \overline{RO} are also perpendicular bisectors of each other. $HO = MO$ and $RO = RO$. Thus the four right triangles are congruent by SAS.

3. The diagonals of the rhombus may be used to find the area of the rhombus. The diagonals of the rhombus divide it into four congruent right triangles. Therefore, the area of the rhombus is four times the area of any one of the right triangles. The area of a right triangle can be found by taking one-half the product of its legs. One leg is one-half the length of one diagonal of the rhombus (diagonal 1). The other leg is one-half the length of the other diagonal of the rhombus (diagonal 2).

$$\text{Area of Rhombus} = 4 \left(\frac{1}{2} \right) \left(\frac{\text{diagonal 1}}{2} \right) \left(\frac{\text{diagonal 2}}{2} \right)$$

so

$$\text{Area of Rhombus} = \frac{1}{2}(\text{diagonal 1})(\text{diagonal 2})$$

(G.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student is expected to:

(A) use constructions to explore attributes of geometric figures and to make conjectures about geometric relationships; and

(G.3) **Geometric structure.**

The student applies logical reasoning to justify and prove mathematical statements.

The student is expected to:

(B) construct and justify statements about geometric figures and their properties;

Connections to TAKS:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Extension Questions:

- What characteristics of a rhombus helped you decide how to construct RHOM?

A rhombus is a parallelogram with four congruent sides.

Responses would vary from this point on. The following response represents the construction approach used in the sample solution:

This leads to the diagonals of the rhombus being perpendicular bisectors of each other. Therefore, start by constructing a segment and its perpendicular bisector. Then decide the length you want the rhombus' sides to be and locate points on the perpendicular bisector of your segment to complete the rhombus.

- How does a square compare with a rhombus?

A square is a rhombus with congruent diagonals. You could also say it is a rhombus with adjacent angles being right angles.

- How does a kite compare with a rhombus?

A kite is a quadrilateral with perpendicular diagonals. It has to have two pairs of adjacent sides that are congruent, but it does not necessarily have congruent opposite sides.

A kite does not have to be a parallelogram.

Tiling with Four Congruent Triangles

Shannon is helping her mom decide on triangular tile patterns for the kitchen they are going to remodel. Using colored construction paper cutouts, they saw they could easily use right triangles (isosceles or scalene), since two congruent right triangles form a rectangle. Shannon began to wonder about using congruent acute or congruent obtuse triangles, which would give a very different look. This led to her question:

1. Can four congruent triangles always be arranged to form a triangle?
Describe how you would explore this question.
2. If your response to # 1 is no, give a counter-example.
3. If your response to # 1 is yes, state a conjecture about how the original triangle and its three copies are related to the newly formed triangle.
4. If your response to # 1 is yes, write a proof that uses transformations to justify your conjecture.



Notes

Materials:

Construction paper, patty paper, unlined paper, and construction tools (including geometry software)

Geometry TEKS Focus:**(G.3) Geometric structure.**

The student applies logical reasoning to justify and prove mathematical statements.

The student is expected to:

(B) construct and justify statements about geometric figures and their properties;

(G.10) Congruence and

the geometry of size. The student applies the concept of congruence to justify properties of figures and solve problems.

The student is expected to:

(A) use congruence transformations to make conjectures and justify properties of geometric figures including figures represented on a coordinate plane; and

Additional Geometry TEKS:

(G.2) Geometric structure. The student analyzes geometric relationships in order to make and verify conjectures.

The student is expected to:

(B) make conjectures about angles, lines, polygons, circles, and three-dimensional figures and determine the validity of the conjectures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

Teacher Notes

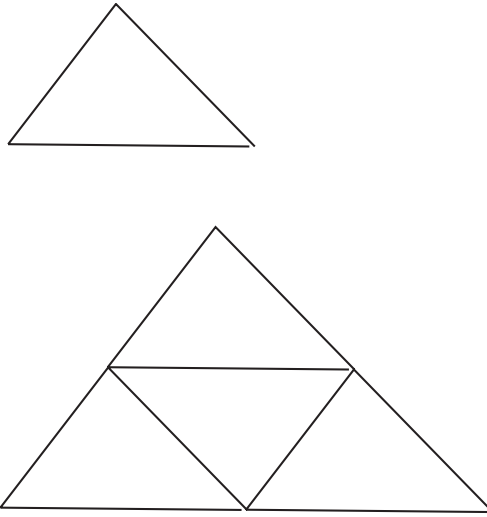
Scaffolding Questions:

- How can you arrange the triangles so that sides are collinear?
- How do you know that the sides of the smaller triangles are collinear?
- After you have arranged the four triangles, where are their vertices located?
- One triangle could be described as an inner triangle. What can you say about its vertices?
- As you arrange the four triangles, think about transformations. What transformations are you using?
- How can you label information on the triangles to make it easier for you to describe these transformations?
- Recall the definition of a triangle midsegment. Recall the triangle midsegment properties we investigated in class. How does this activity relate to these investigations?

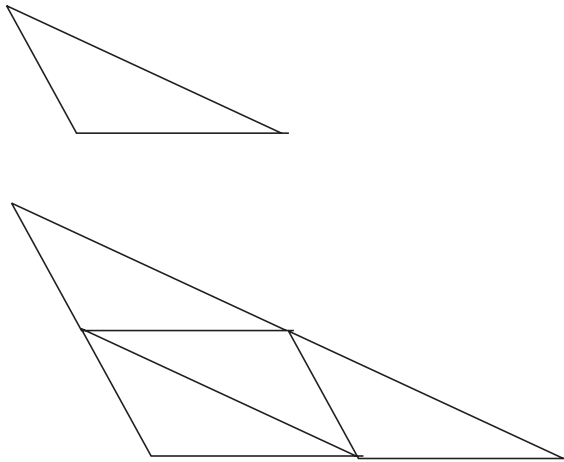
Sample Solutions:

1. To explore this question, cut out four copies of an acute triangle using different colors for each triangle. Arrange the triangles so that they do not overlap and they form a triangle. Repeat this using four copies of an obtuse triangle. You could also draw a triangle on patty paper and make three copies. Then arrange the four triangles to form the larger triangle.

Acute triangle example:



Obtuse triangle example:



2. Since a larger triangle may be formed with the four triangles, there is no apparent counterexample.
3. In both cases the four congruent triangles form a larger triangle that is similar to the original triangle and its copies.

The sides of the larger triangle are twice as long as the corresponding sides of the original triangle. The area of the large triangle is four times the area of the original triangle because it is made up of the four non-overlapping congruent triangles.

(G.3) Geometric structure.
The student applies logical reasoning to justify and prove mathematical statements.

The student is expected to:
(C) use logical reasoning to prove statements are true and find counter examples to disprove statements are false;

(G.5) Geometric patterns.
The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to:
(C) use properties of transformations and their compositions to make connections between mathematics and the real world, such as tessellations; and

(G.9) Congruence and the geometry of size. The student analyzes properties and describes relationships in geometric figures.

The student is expected to:
(B) formulate and test conjectures about the properties and attributes of polygons and their component parts based on explorations and concrete models;

(G.10) **Congruence and the geometry of size.** The student applies the concept of congruence to justify properties of figures and solve problems.

The student is expected to:

(B) justify and apply triangle congruence relationships.

(G.11) **Similarity and the geometry of shape.** The student applies the concept of similarity to justify properties of figures and solve problems.

The student is expected to:

(A) use and extend similarity properties and transformations to explore and justify conjectures about geometric figures;

If the large triangle is similar to the original triangle, the ratio of the area of the large triangle to the area of the original triangle is the square of the ratio of the corresponding sides.

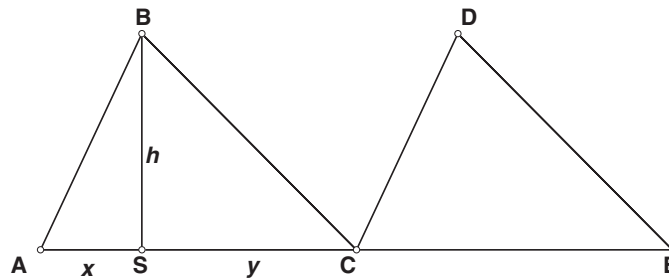
$$\frac{\text{Area of large triangle}}{\text{Area of original triangle}} = \left(\frac{2}{1}\right)^2 = \frac{4}{1}$$

4. Transformational Proof:

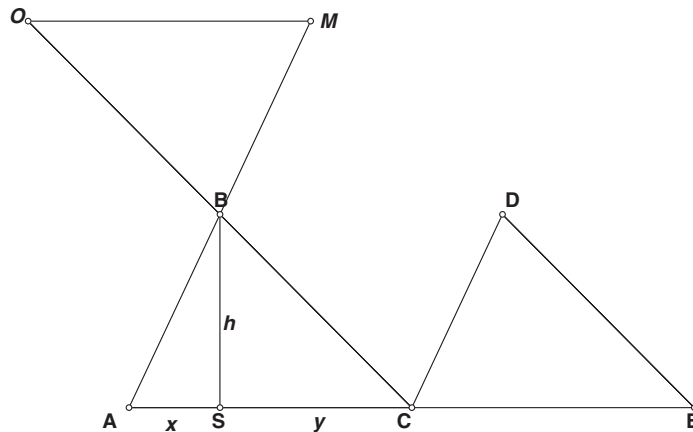
Step 1: Draw $\triangle ABC$ so that segment \overline{AC} is horizontal. Draw altitude \overline{BS} .

Let $BS = h$, $AS = x$, and $SC = y$.

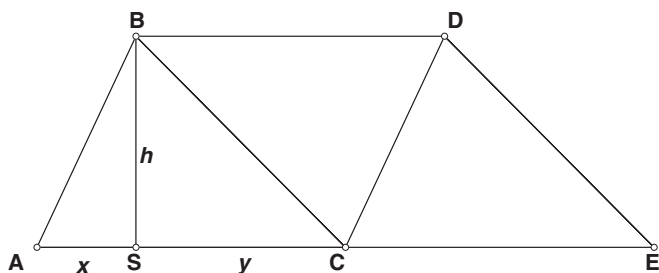
Step 2: Translate $\triangle ABC$ $x + y$ units to the right to form $\triangle CDE \cong \triangle ABC$. A , C , and E are collinear.



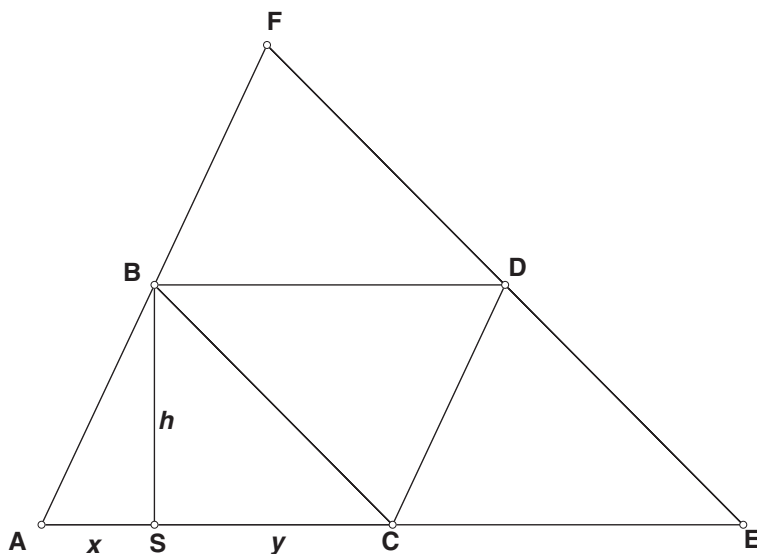
Step 3: Rotate $\triangle ABC$ 180° counterclockwise about point B to form $\triangle MBO \cong \triangle ABC$.



Step 4: Translate $\triangle BMO$ y units to the right and h units down to form $\triangle DCB \cong \triangle ABC$.



Step 5: Translate $\triangle ABC$ x units to the right and h units up to form $\triangle BFD \cong \triangle ABC$. $A, B,$ and F are collinear. $F, D,$ and E are collinear.



The four triangles are congruent. The three angles of the smaller triangle are congruent to the three corresponding angles of the larger triangle.

$$\angle A \cong \angle A, \angle ABC \cong \angle F, \angle ACB \cong \angle E$$

The large triangle is similar to the original triangle by AAA similarity.

$$\triangle AFE \sim \triangle ABC$$

Connections to TAKS:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Extension Questions:

- In this activity, what have you shown to be true?

Triangles of any shape can be used to tile a plane because the four congruent triangles can be arranged in a non-overlapping way to form a larger triangle.

- How are the vertices of the four original triangles related to the vertices of the larger triangle that is formed?

Three of the triangles may be considered to be outer triangles, and the fourth triangle is inside the larger triangle. One vertex from each of the outer triangles is a vertex of the larger triangle.

The vertices of the inner triangle are on the sides of the larger triangle and will become the midpoints of the sides of the larger triangle. Each side of the large triangle is formed by two corresponding sides of the congruent triangles. Therefore, each side of the large triangle is twice as long as a side of the original triangle.

- How does this result relate to the triangle midsegment theorems?

It is the converse of the following theorem:

If the midsegments of a triangle are drawn, then four triangles are formed, and the four triangles are congruent.

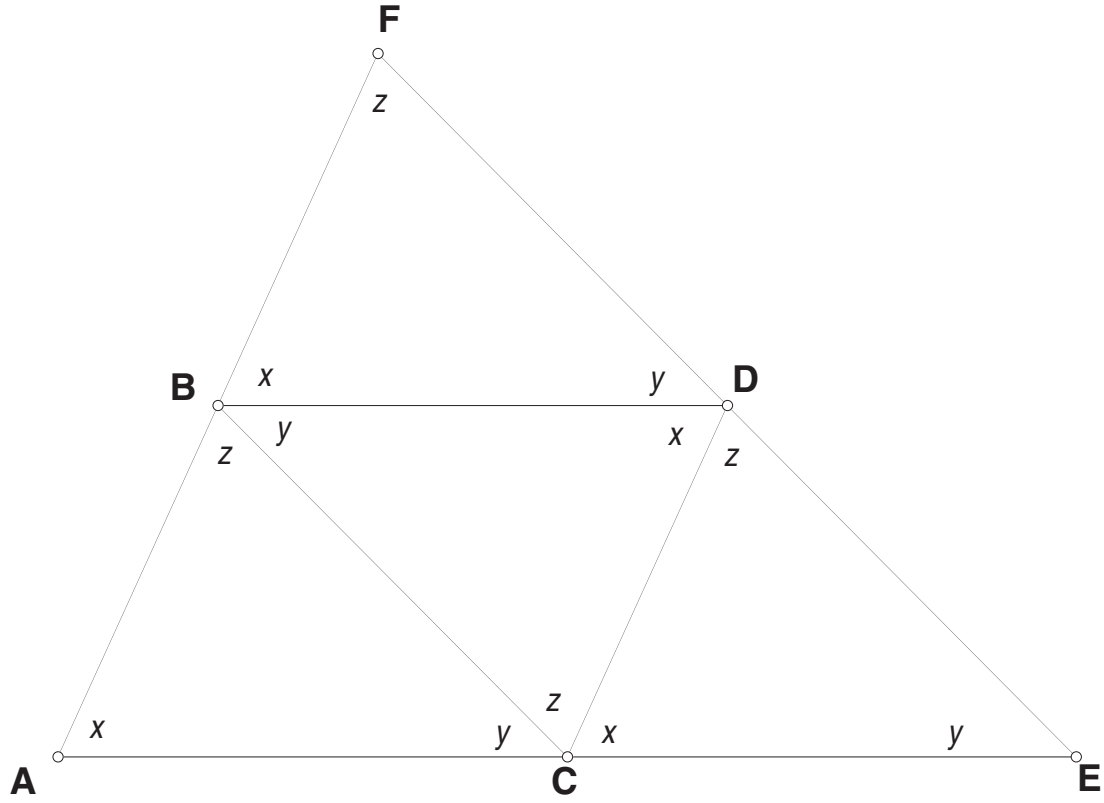
- How can you prove the result of this activity using algebraic thinking?

To be sure the four congruent triangles are forming a triangle, we need to show that points A, B, and F are collinear; points F, D, and E are collinear; and points C, E, and A are collinear.

Since $\triangle ABC \cong \triangle DCB \cong \triangle CDE \cong \triangle BFD$, let the measures of the congruent corresponding angles be x , y , and z , as shown on the diagram.

We know that the sum of the angles of a triangle equal 180° . Therefore, we know that $x + y + z = 180$.

This shows that B is collinear with A and F, D is collinear with F and E, and C is collinear with E and A. Therefore, the four congruent triangles form the large triangle with no overlap.



Student Work Sample

As an extension to the task, a student was asked to justify his conjecture using a coordinate proof. The student started with and labeled the coordinates. A more complete proof would have included an explanation of why he chose the coordinates of the other points on his diagram. The student used E and G interchangeably from time to time.

Some of the solution guide criteria exemplified by this work are the following:

- Demonstrates geometric concepts, processes, and skills.

The student shows the correct use of the distance formula. He correctly simplifies the radicals and determines the ratio of corresponding sides in the smaller and larger triangles.

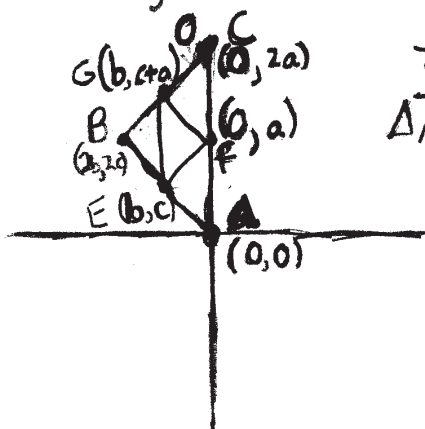
- Uses geometric and other mathematical principles to justify the reasoning used to analyze the problem.

The student correctly uses the distance formula to get the length of each side of the triangles. Also, he compares the length of the sides to get the ratio to prove similarity of the triangles.

- Communicates a clear, detailed, and organized solution strategy.

The student states his conjecture that the smaller triangles are similar to the larger triangle. He indicates that he will use the distance formula. He shows all the necessary steps in using the distance formula and in simplifying the ratios of the lengths of the corresponding sides. He states his conclusion that the triangles are similar and gives a reason for this conclusion.

Tiling with 4 Congruent Triangles



I must prove that $\triangle CEF$, $\triangle AEF$, $\triangle BEG$, and $\triangle EFG$ are similar to $\triangle ABC$. Using the distance formula to do this,

$$\overline{AC} = \sqrt{0^2 + (2a)^2} \text{ or } \sqrt{4a^2} \text{ or } 2a$$

$$\overline{CE} = \sqrt{0^2 + a^2} \text{ or } \sqrt{a^2} \text{ or } a$$

$$\overline{AF} = \sqrt{0^2 + a^2} \text{ or } a$$

$$\overline{EG} = \sqrt{(b-b)^2 + (c-c+a)^2} \text{ or } \sqrt{0^2 + a^2} \text{ or } a$$

$$\frac{\overline{CE}}{\overline{AC}} = \frac{\overline{AF}}{\overline{AC}} = \frac{\overline{EG}}{\overline{AC}} = \frac{a}{2a} = \frac{1}{2}$$

$$\overline{AB} = \sqrt{(b)^2 + (c)^2} \text{ or } \sqrt{b^2 + c^2} \text{ or } \sqrt{b^2 + c^2}$$

$$\overline{AG} = \sqrt{b^2 + c^2}$$

$$\overline{BG} = \sqrt{(b-b)^2 + (c-c)^2} \text{ or } \sqrt{0^2 + 0^2} \text{ or } 0$$

$$\overline{EF} = \sqrt{b^2 + (c-a-a)^2} \text{ or } \sqrt{b^2 + c^2}$$

$$\frac{\overline{AG}}{\overline{AB}} = \frac{\overline{BG}}{\overline{AB}} = \frac{\overline{EF}}{\overline{AB}} = \frac{\sqrt{b^2 + c^2}}{\sqrt{b^2 + c^2}} = 1$$

$$\overline{BC} = \sqrt{(b-0)^2 + (c-2a)^2} \text{ or } \sqrt{b^2 + 4c^2 - 8ac + 4a^2} \text{ or } \sqrt{b^2 + c^2 - 2ac + a^2}$$

$$\overline{BE} = \sqrt{(b-b)^2 + (c-c+a)^2} \text{ or } \sqrt{0^2 + (c-a)^2} \text{ or } \sqrt{b^2 + c^2 - 2ac + a^2}$$

$$\overline{CE} = \sqrt{b^2 + (2a-a+c)^2} \text{ or } \sqrt{b^2 + (a+c)^2} \text{ or } \sqrt{b^2 + c^2 - 2ac + a^2}$$

$$\overline{FG} = \sqrt{b^2 + (c-a)^2} \text{ or } \sqrt{b^2 + c^2 - 2ac + a^2}$$

$$\frac{\overline{BE}}{\overline{BC}} = \frac{\overline{CE}}{\overline{BC}} = \frac{\overline{FG}}{\overline{BC}} = \frac{\sqrt{b^2 + c^2 - 2ac + a^2}}{\sqrt{b^2 + c^2 - 2ac + a^2}} = 1$$

This means $\triangle CEF$, $\triangle AEF$, $\triangle BEG$, $\triangle EFG$, are congruent and similar to $\triangle ABC$ because the corresponding sides are 1:2.



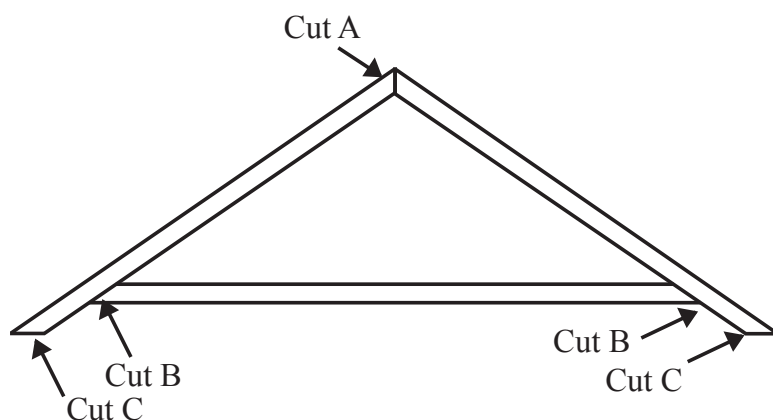
Shadow's Doghouse

You are planning to build a new doghouse for your German Shepherd, Shadow. The doghouse will be a rectangular solid: 24 inches across the front, 36 inches deep, and 30 inches high. The cross section of the pitched roof is an isosceles triangle. The distance from the floor of the doghouse to the peak of the roof is 39 inches.

There will be four roof trusses (shown below), which are isosceles triangles, to support the pitched roof.

To determine the materials you need to purchase and how you will construct the frame, you make careful plans on paper before you begin construction.

1. Make a sketch of the doghouse showing the front, side, and roof.
2. How could you ensure the four roof trusses are precisely the same shape and size using the least number of measurements? Describe at least two ways. What geometry concepts are being used?
3. The roof trusses are to be cut from 1-inch by 2-inch boards, as diagrammed below:

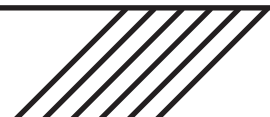


At what angles should cuts A, B, and C be made? What geometry concepts are being used?

4. The roof is to have a 3-inch overhang. That means that there will be a 3-inch extension of the roof at the front, back, and sides. The roof is to be made from two pieces of plywood and covered with shingles. The

shingles are laid so that they overlap, with the exposed (visible) area of a shingle measuring 6 inches by 6 inches. How many shingles are needed? Explain how you determine this, citing the geometry concepts you use.

5. The roof, sides, front, back, and floor of the doghouse will be made of plywood, which is available in 4-foot by 8-foot sheets and half-sheets (2 feet by 8 feet or 4 feet by 4 feet). The opening on the front of the doghouse will be 12 inches wide by 18 inches high. The front and back pieces are to be cut as pentagons, not as rectangles with an added triangle. How many sheets and/or half-sheets of plywood are needed for the entire doghouse, and how would you lay out the pieces to be cut? What geometry concepts are being used to ensure opposite walls are precisely the same size?
6. After estimating your total costs, you decide to consider making Shadow's doghouse smaller, with a floor 20 inches wide and 30 inches deep. The shape of the new doghouse would be similar to the original doghouse. A dog that is the size of Shadow requires a doghouse with a floor area of at least 4 square feet and a capacity of at least 9 cubic feet. Would the smaller doghouse be large enough for Shadow? Explain.





Notes

Materials:

One ruler, calculator, and protractor per student
Unlined paper for drawing

Geometry TEKS Focus:**(G.8) Congruence and the geometry of size.**

The student uses tools to determine measurements of geometric figures and extends measurement concepts to find perimeter, area, and volume in problem situations.

The student is expected to:

(A) find areas of regular polygons, circles, and composite figures;

(D) find surface areas and volumes of prisms, pyramids, spheres, cones, cylinders, and composites of these figures in problem situations.

(G.11) **Similarity and the geometry of shape.** The student applies the concept of similarity to justify properties of figures and solve problems.

The student is expected to:

(D) describe the effect on perimeter, area, and volume when one or more dimension of a figure are changed and apply this idea in solving problems.

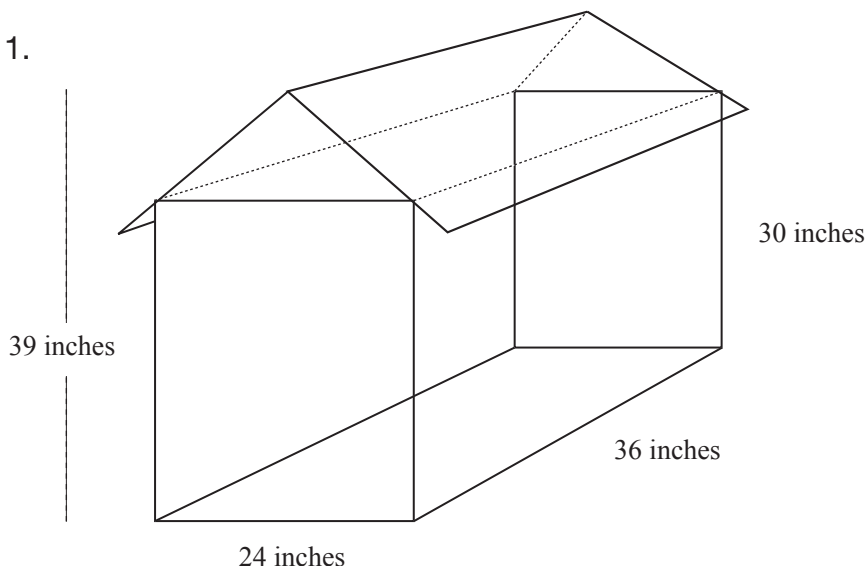
Teacher Notes

Scaffolding Questions:

- What does it mean for geometric figures to be congruent?
- What are the possible ways to prove triangles are congruent?
- What would be reasonable ways to build the roof trusses so that they are congruent triangles?
- How can Pythagorean Triples help you determine measurements in this problem?
- What are the angle characteristics for various triangles?
- What special triangles are formed by the roof trusses?
- What do you know about the altitude to the base of an isosceles triangle?
- What are the shape and dimensions of the plywood pieces that will form the roof?
- How can you plan the plywood pieces you will need to cut for the doghouse?
- If figures are similar, what do you know about the ratio of the corresponding linear measurements, the ratio of areas, and the ratio of volumes?

Sample Solutions:

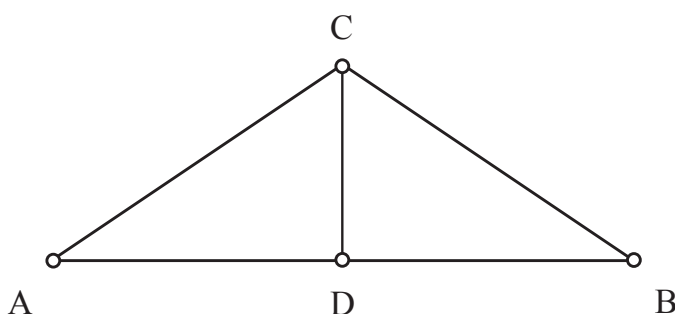
1.



2. For the triangular roof trusses to be the same shape and size, we must design and build them so that we have congruent triangles. Triangles are congruent by SSS, SAS, ASA, or SAA. The most reasonable congruence relationship to use in building the roof trusses is probably SSS. We would determine how long the boards need to be and where they will be joined. Then we would cut the boards.

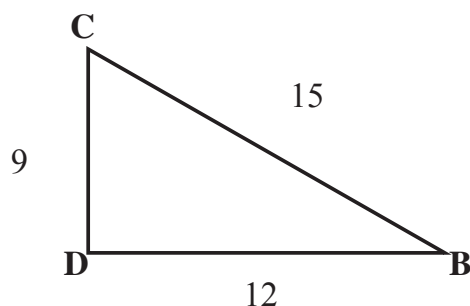
We know the beams will be 24 inches across and 9 inches high.

$AB = 24$, $CD = 9$, \overline{CD} is perpendicular to \overline{AB} , $AC = CB$



Since the beams form an isosceles triangle, we know that the sides that will support the roof are congruent, and that the altitude segment \overline{CD} to the base \overline{AB} bisects \overline{AB} .

Using right triangles and Pythagorean Triples, we find the lengths of the congruent sides. Segments \overline{AC} and \overline{BC} are 15 inches.



Measuring the angles with a protractor, we find that $m\angle A = m\angle B \approx 37^\circ$ and $m\angle ACB \approx 106^\circ$.

If we build the triangular trusses to be congruent by SSS, we would cut four 24-inch boards and eight 18-inch boards. Three inches of each board is for the roof overhang.

Additional Geometry TEKS:

(G.1) Geometric structure.

The student understands the structure of, and relationships within, an axiomatic system.

The student is expected to:

(A) develop an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems;

(G.6) Dimensionality and the geometry of location.

The student analyzes the relationship between three-dimensional geometric figures and related two-dimensional representations and uses these representations to solve problems.

The student is expected to:

(C) use orthographic and isometric views of three-dimensional geometric figures to represent and construct three-dimensional geometric figures and solve problems.

(G.10) **Congruence and the geometry of size.** The student applies the concept of congruence to justify properties of figures and solve problems.

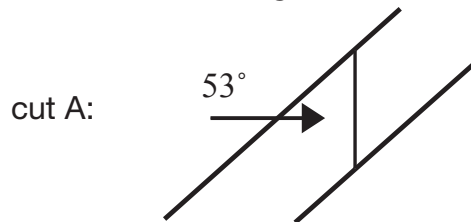
The student is expected to:

(A) use congruence transformations to make conjectures and justify properties of geometric figures including figures represented on a coordinate plane; and

(B) justify and apply triangle congruence relationships.

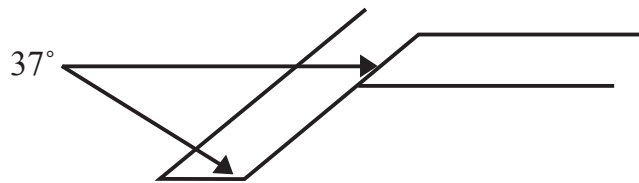
If we build the triangular trusses to be congruent by ASA, we would cut the four 24-inch boards and then measure the base angles on each end to be 37 degrees.

- Since the vertex angle of the isosceles triangle formed by the outer edges of the boards measures 106 degrees, and the angled edge lies along the altitude to the base of the isosceles triangle, cut one needs to be made at an angle of 53 degrees. This is because the altitude from the vertex angle of an isosceles triangle is also the bisector of that angle.



The top and bottom edges of the baseboard of the truss (the base of the isosceles triangle) and the angled edge of the bottom of the boards that are the sides of the beam are parallel. Since corresponding angles formed by transversals on parallel lines are congruent, cuts two and three need to be made at angles of 37°.

cuts B and C:



- To determine the number of shingles needed, we compute the surface area to be covered. The surface area is $42(18) = 756$ square inches. The number of shingles needed for one side is

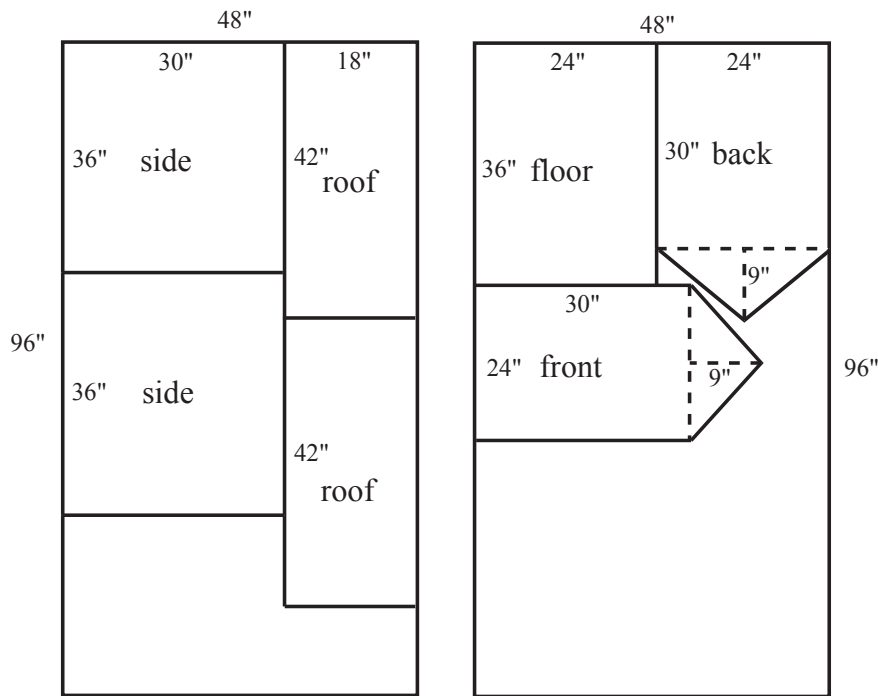
$$\frac{756 \text{ square inches}}{36 \text{ square inches for one shingle}} = 21 \text{ shingles}$$

- The total square area of plywood needed for the doghouse is

$2[30(36) + 18(42) + 24(30) + 0.5(9)(24)] + 24(36) = 6,192$ square inches.

One full sheet of plywood has an area of $48(96) = 4,608$ square inches. One and one-half sheets would provide an area of 6,912 square inches. However, the layout of the pieces (side, front, back, roof, and floor) must be considered.

The following possible layout shows that two full pieces of plywood are needed to efficiently cut the pieces as planned.



The sides are congruent rectangles, and the front and back are congruent pentagons since corresponding sides and corresponding angles are congruent.

- The floor space of the new doghouse would be 20 inches by 30 inches, or 600 square inches. Convert to square feet.

$$600 \text{ in}^2 \cdot \frac{1 \text{ ft}^2}{144 \text{ in}^2} = 4.167 \text{ ft}^2$$

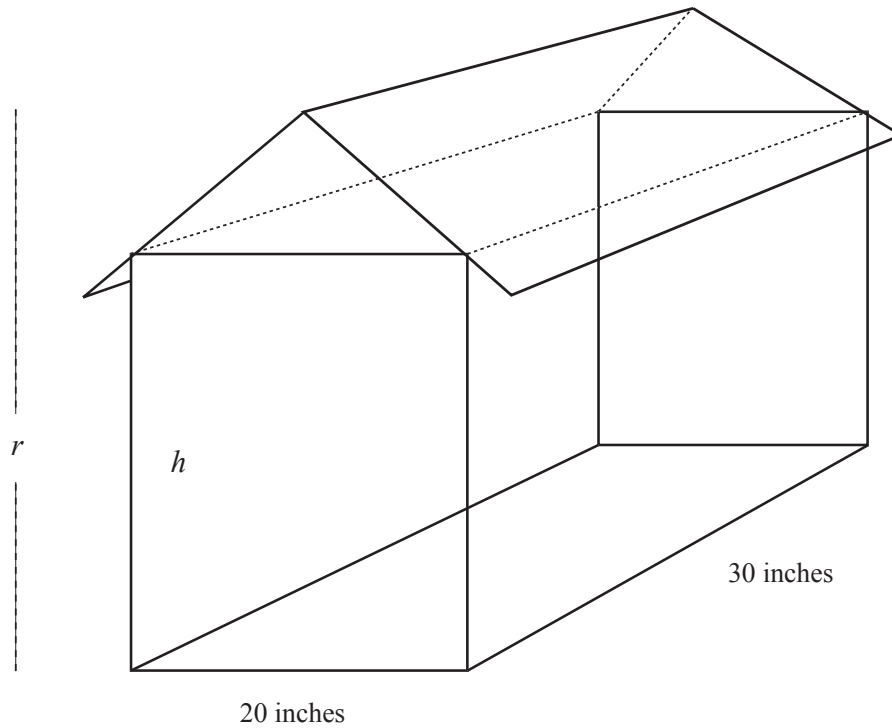
The floor space is more than 4 ft².

Since the two doghouses are similar in shape, the other dimension of the doghouse may be determined by using ratios.

Connections to TAKS:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.



If the shapes are similar, then

$$\frac{h}{20} = \frac{30}{24} \quad \text{and} \quad \frac{r}{20} = \frac{39}{24}$$

$$h = 20\left(\frac{30}{24}\right) = 25 \text{ in} \quad r = 20\left(\frac{39}{24}\right) = 32.5 \text{ in}$$

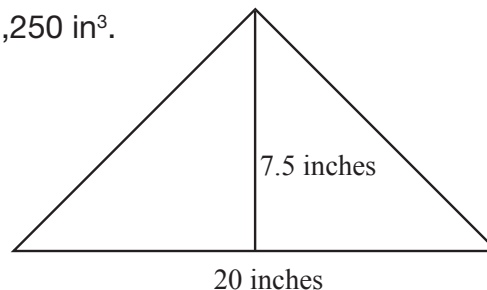
The capacity of the original doghouse is the volume of the rectangular prism portion plus the volume of the triangular prism of the roof.

The volume of the rectangular prism is 20 inches by 30 inches by 25 inches, or $15,000 \text{ in}^3$.

$$15,000 \text{ in}^3 \cdot \frac{1 \text{ ft}^3}{12 \text{ in} \cdot 12 \text{ in} \cdot 12 \text{ in}} = 8.68 \text{ ft}^3$$

The volume of the triangular prism is the area of the triangle times the length. The height of the triangle is $32.5 - 25 = 7.5$ inches.

The volume of the prism is $75(30) = 2,250 \text{ in}^3$.



$$\text{Area of the triangle} = \frac{1}{2} 7.5(20) = 75 \text{ in}^2$$

The volume of the new doghouse would be $8.68 + 1.30 \text{ ft}^3$, or 9.98 ft^3 .

$$2,250 \text{ in}^3 \cdot \frac{1 \text{ ft}^3}{12 \text{ in} \cdot 12 \text{ in} \cdot 12 \text{ in}} = 1.30 \text{ ft}^3$$

This doghouse would be large enough for Shadow.

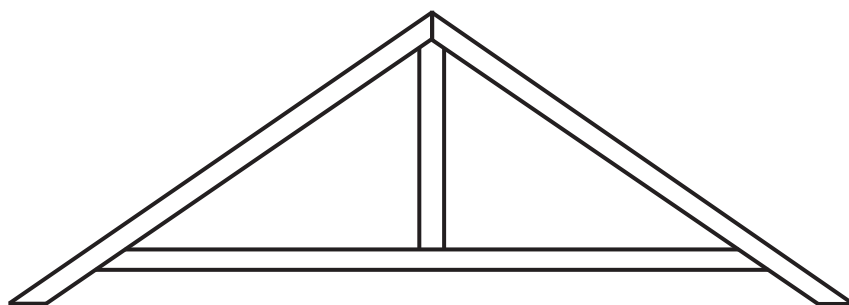
Extension Questions:

- Build a scale model of Shadow’s doghouse.

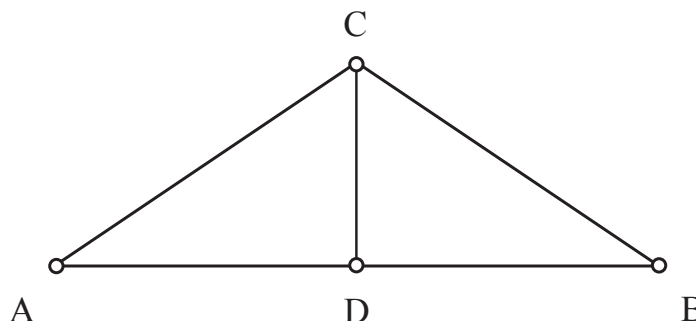
Answers will vary.

- Describe how you could build the roof trusses using congruent scalene triangles.

Assuming the peak of the roof will be centered on the doghouse, each truss would consist of two congruent scalene triangles as shown below:



The truss would consist of the 24-inch base, a 9-inch board that is perpendicular to the base, and the 15-inch roof supports. This is using two scalene triangles that are congruent by SSS:



$AD = BD = 12$, $CD = 9$ and $AC = BC = 15$.

For the remaining questions, suppose that you decide to build the doghouse so that the floor is a regular hexagon.

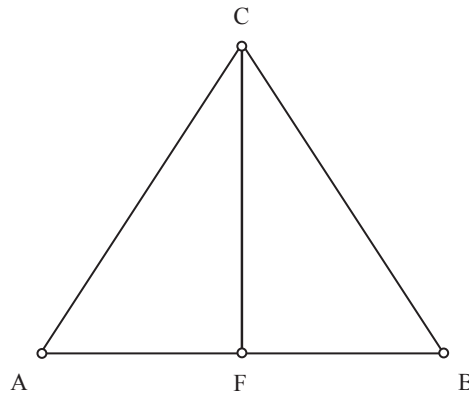
- Determine the dimensions, to the nearest inch of the hexagon, and describe how you will draw the hexagon on the plywood in order to cut it. Remember that you need at least 4 square feet of floor space.

The hexagon will consist of six congruent equilateral triangles. To determine the side of the equilateral triangle, solve the following area problem:

Area of polygon:

One-half the apothem of the polygon times its perimeter = one-half times the altitude of the triangle space times six times the base of the triangle.

Since the apothem is the altitude of the equilateral triangle, and the altitude divides the equilateral triangle into two 30-60-90 triangles, use the Pythagorean Theorem to express the altitude, a , in terms of a side, s , of the equilateral triangle.



$\triangle ABC$ is an equilateral triangle, and \overline{CF} is the altitude to \overline{AB} .

Let $CF = a$, and $BC = s$. Then $FB = \frac{s}{2}$, and

$$\begin{aligned} a^2 + \left(\frac{s}{2}\right)^2 &= s^2 \\ a^2 &= s^2 - \frac{s^2}{4} \\ &= \frac{3s^2}{4} \\ \text{so that } a &= \frac{\sqrt{3}}{2} s. \end{aligned}$$

Next, get an expression for the area of the hexagon:

$$\begin{aligned} A &= \frac{1}{2}ap \\ &= \frac{1}{2}a(6s) \\ &= 3\left(\frac{\sqrt{3}}{2}s\right)s \\ &= \frac{3\sqrt{3}}{2}s^2 \end{aligned}$$

Now

$$\begin{aligned} \text{let } \frac{3\sqrt{3}}{2}s^2 &= 4 \\ 3\sqrt{3}s^2 &= 8 \\ s^2 &= \frac{8}{3\sqrt{3}} \\ &\approx 1.54 \text{ sq. ft.} \end{aligned}$$

$$\begin{aligned} \text{so that } s &\approx 1.24 \text{ ft} \\ &= 14.88 \text{ inches} \end{aligned}$$

Rounding to the nearest inch, we need to cut a regular hexagon with a side that is 15 inches.

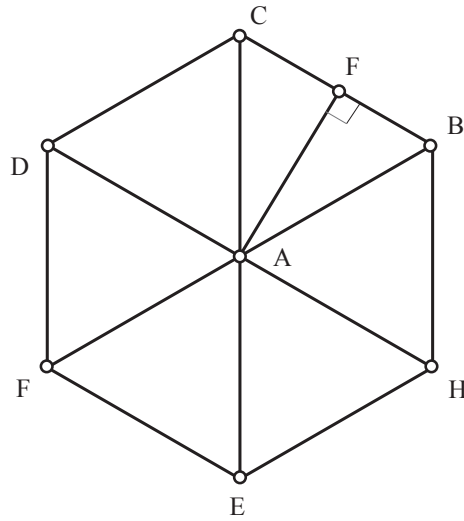
One way to draw the hexagonal floor is to draw a circle of radius 15 inches. Then mark off and draw on the circle chords that are 15 inches long.

- Assume the floor-to-roof edge must still be 30 inches, and the door opening must be at least 12 inches wide by 18 inches high. How much plywood will be needed for the sides?

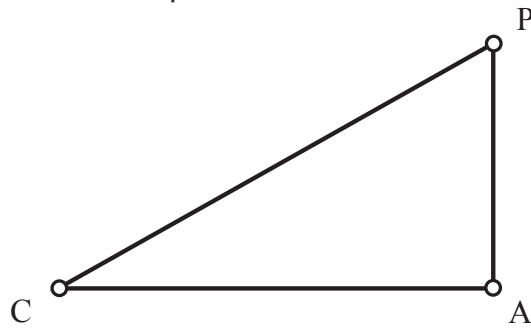
You would need to cut six congruent rectangles that are 15 inches wide by 30 inches high and then cut the opening in one of them. One 4-foot by 8-foot piece of plywood would be enough, measuring 15 inches across the 4-foot width twice and 30 inches along the 8-foot length three times.

- The roof for the redesigned doghouse will be a hexagonal pyramid with a height (altitude) of 9 inches. Describe how you would design the beams to support the roof and the plywood sections that will form the roof.

The base of the hexagonal pyramid that will be the roof will be congruent to the floor of the doghouse. The base is shown below.

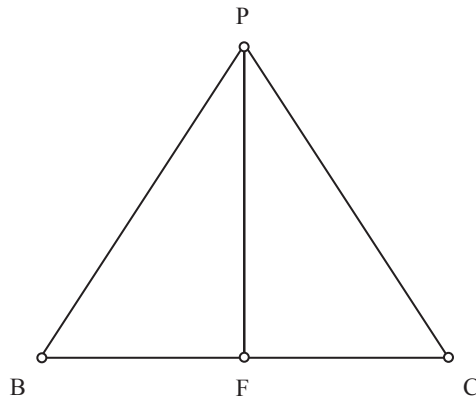


Six congruent triangular roof trusses can be made. Let P be the point at the top of the roof. The triangle CAP is a sample roof truss.



$CA = 15$ inches, $PA = 9$ inches, and $CP = 17.49$ inches (approximately) by the Pythagorean Theorem. (The board corresponding to segment PA needs to be a hexagonal post, and it is a common leg of all six right triangles forming the roof trusses.)

The six congruent plywood triangles that would form the roof would look like the following triangle:



$BC = 15$ inches. $BP = PC = 17.49$ inches, and the segments are the lateral edges of one of the six congruent triangle faces of the pyramid that forms the roof.

- How will the capacity (volume) of the new doghouse compare with that of the original doghouse?

The capacity of the new doghouse will be the volume of the regular hexagonal prism that is Shadow's room plus the volume of the hexagonal pyramid that is the roof:

Prism Volume = height times base area

Pyramid Volume = one-third height times base area.

Base Area:

$$\text{Find the apothem of the hexagon: } a = \frac{\sqrt{3}}{2} s = \frac{\sqrt{3}}{2} \cdot 15 \approx 12.99$$

$$A = \frac{1}{2} ap = 0.5(12.99)(6 \cdot 15) = 584.55 \text{ square inches}$$

Prism Volume:

$$V = 30 \cdot 584.55 = 17536.5 \text{ cubic inches}$$

Pyramid Volume:

$$V = \frac{1}{3} \cdot 9 \cdot 584.55 = 1753.65 \text{ cubic inches}$$

Total Volume:

$$V = 17536.5 + 1753.65 = 19290.15 \text{ cubic inches}$$

Convert to cubic feet.

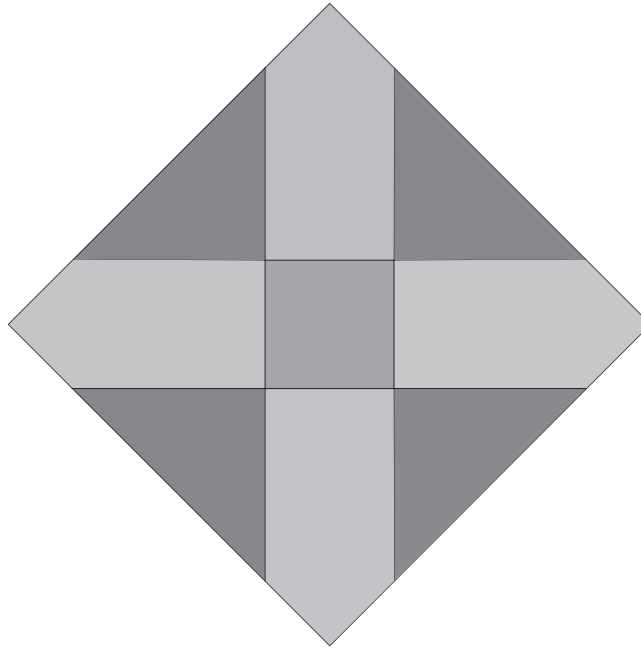
$$19290.15 \text{ cubic inches} \cdot \frac{1 \text{ cubic foot}}{12 \text{ cubic inches}} = 11.16 \text{ cubic feet}$$

The volume is 11.16 cubic feet.



The School Flag

The plan for a high school's flag displays a cross on a square as shown below:



The square in the center of the cross will be silver, the rest of the cross will be red, and the remainder of the flag is to be black.

The total area of the flag will be 400 square inches, and the cross should fill 36% of the square flag.

The total cost of a flag is based on the parts: silver (most expensive), red (next in cost), and black (least expensive). These cost issues prompt the question:

What percentage of the area of the flag is each colored section?

Explain your reasoning.



Notes

Materials:

One calculator per student
 Construction paper in different colors, unlined paper, and construction tools (including geometry software)

Geometry TEKS Focus:**(G.8) Congruence and the geometry of size.**

The student uses tools to determine measurements of geometric figures and extends measurement concepts to find perimeter, area, and volume in problem situations.

The student is expected to:

(A) find areas of regular polygons, circles, and composite figures;

(G.10) Congruence and the geometry of size. The student applies the concept of congruence to justify properties of figures and solve problems.

The student is expected to:

(B) justify and apply triangle congruence relationships.

Teacher Notes

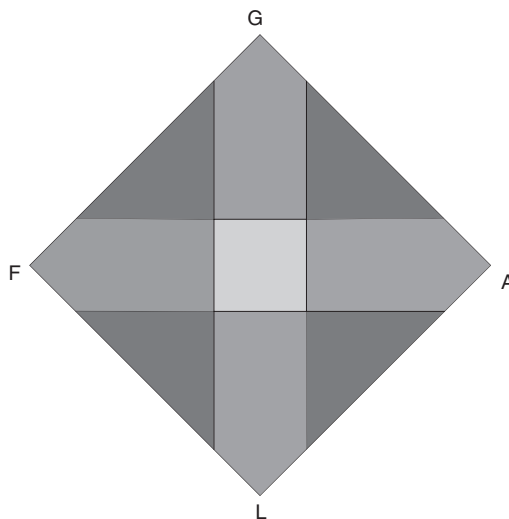
Scaffolding Questions:

- Make a model using different colors of construction paper.
- If you break the flag apart into pieces, what kinds of polygons do you see?
- What are the dimensions of the shapes that could vary?
- What information does the problem give you that restricts the dimensions?
- How could you dissect your model and rearrange the pieces so that you have the same colored pieces grouped together to form rectangles or squares?
- What properties of a square help you find congruent triangles?
- As you rearrange pieces, think about transformations. What transformations are you using?

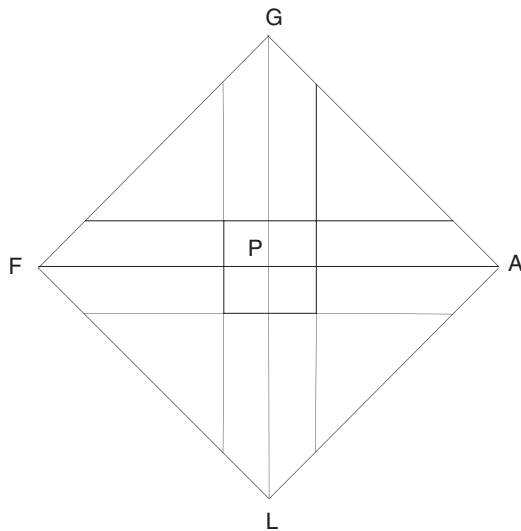
Sample Solutions:

The flag can be cut into pieces, and the pieces can be rearranged so that each color forms a rectangular region.

Label the original square FLAG.

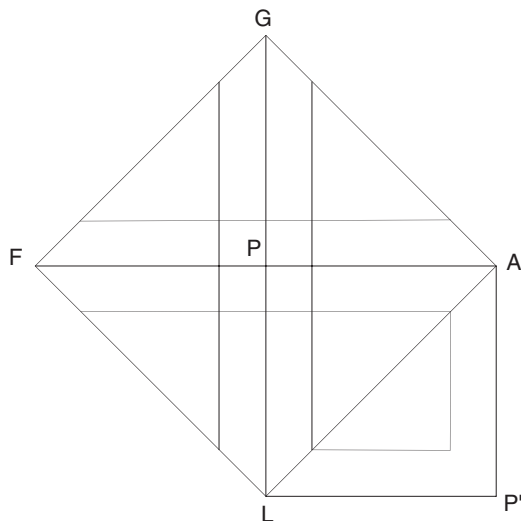


Draw diagonals \overline{FA} and \overline{LG} . Label their point of intersection P.



Since the diagonals of the square are congruent, bisect each other, and are perpendicular, triangles FPL, LPA, APG, and GPF are congruent by SAS.

Translate $\triangle FPG$ so that segment \overline{FG} coincides with segment \overline{LA} to form the following:



Triangles LPA and $LP'A$ form a square that is one-half of the flag. Therefore, the area of square $PLP'A$ is 200 square inches.

Additional Geometry TEKS:

(G.4) Geometric structure.
The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to select an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

(G.5) Geometric patterns.
The student uses a variety of representations to describe geometric relationships and solve problems.

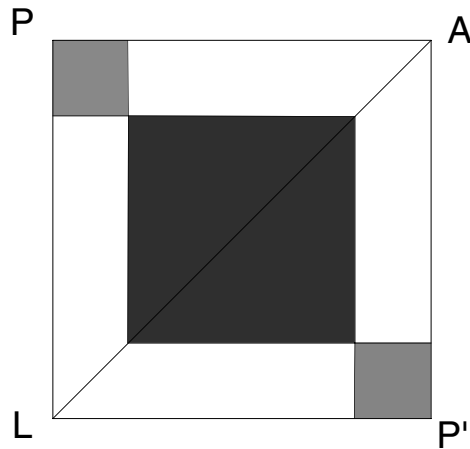
The student is expected to:
(C) use properties of transformations and their compositions to make connections between mathematics and the real world, such as tessellations; and

(G.10) **Congruence and the geometry of size.** The student applies the concept of congruence to justify properties of figures and solve problems.

The student is expected to:

(A) use congruence transformations to make conjectures and justify properties of geometric figures including figures represented on a coordinate plane; and

Now focus on the colors of the regions of square PLP'A:



The original square flag has an area of 400 square inches. Square PLP'A has an area of 200 square inches. If we let s be the side of square PLP'A, then

$$\begin{aligned} s^2 &= 200 \\ s &= 10\sqrt{2} \\ &\approx 14.14 \text{ inches} \end{aligned}$$

Since the cross fills 36% of the flag, the black portion is 64% of the flag. Therefore, in the half-flag the black portion is 64% of 200 square inches, or 128 square inches.

In the figure above, the black portion is square. If we let b be the side of the black square, then

$$\begin{aligned} b^2 &= 128 \\ b &= 8\sqrt{2} \\ &\approx 11.31 \text{ inches} \end{aligned}$$

Now y , the side of the two small silver squares, must be

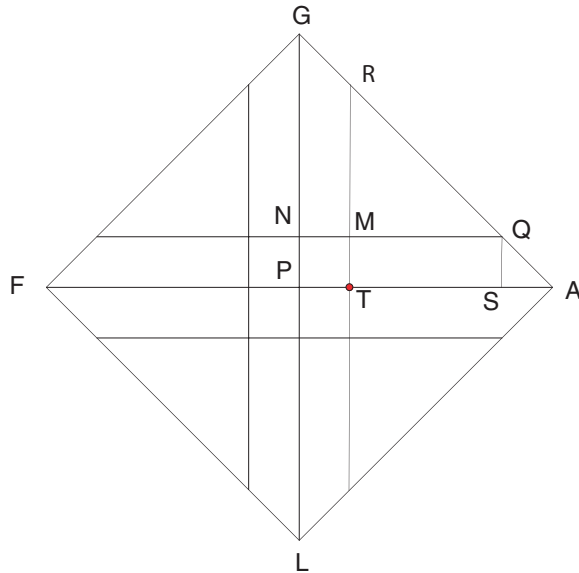
$$y = \frac{14.14 - 11.31}{2} \approx 1.42 \text{ inches}$$

The area of the two small silver squares in the half-flag is $A = 2(1.42)^2 \approx 4$ square inches.

The area of the red portion of the cross in the half-flag must be $200 - 128 - 4 = 68$ square inches.

The color percentages of the half-flag will be the same for the original flag. Therefore, 64% of the flag is to be black, $\frac{68}{200} = 34\%$ is to be red, and $\frac{4}{200} = 2\%$ is to be silver.

Another approach to solving this problem would be to recognize that the variable in this situation is the side of the silver square. The lengths may be expressed in terms of this variable.



\overline{NMTS} is a square. Let the sides of this square be x units. Let \overline{QS} be perpendicular to \overline{AP} . Because parallel lines are equidistant, \overline{QS} is also x units. \overline{SA} would also be x units because $\triangle QAS$ is an isosceles right triangle. $\triangle MTSQ$ is a rectangle, so $\overline{MQ} = \overline{TS}$. As shown in the previous solution, \overline{AP} is $10\sqrt{2}$ units. \overline{MQ} is $10\sqrt{2} - 2x$.

$\triangle RMQ$ is a right isosceles triangle with area

$$\frac{1}{2}(10\sqrt{2} - 2x)(10\sqrt{2} - 2x)$$

The area of the four black triangles is

$$A = 4 \cdot \frac{1}{2}(10\sqrt{2} - 2x)(10\sqrt{2} - 2x)$$

$$A = 2(10\sqrt{2} - 2x)^2$$

As indicated earlier, since the cross fills 36% of the flag, the black portion is 64% of the flag. Therefore, the black

Connections to TAKS:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

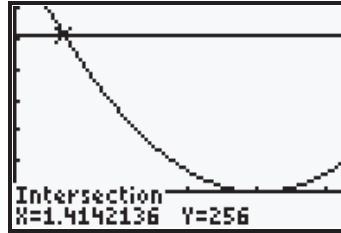
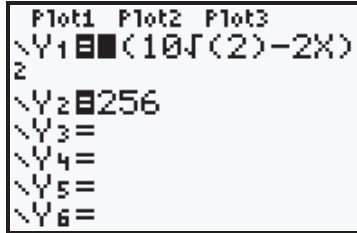
Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

portion is 64% of 400 square inches, or 256 square inches.

$$A = 2(10\sqrt{2} - 2x)^2$$

$$256 = 2(10\sqrt{2} - 2x)^2$$

This equation may be solved using a graphing calculator. The area function may be graphed or a table of values may be found.



X	Y1	Y2
1.4	257.29	256
1.41	256.38	256
1.42	255.48	256
1.43	254.57	256
1.44	253.67	256
1.45	252.77	256
1.46	251.87	256

X=1.41

x is 1.41.

The area of the silver square is $(2x)^2$, or 7.9524 square inches.

The percentage of the total area of 400 is $\frac{7.9524}{400} \approx 0.02$ or 2% of the total area. The red area must be 36% minus 2%, or 34%.

Extension Questions:

- What transformations could you use to rearrange the pieces of the flag to place same-colored pieces together?

Only horizontal and vertical translations of the triangular piece FPL were needed.

We translated a distance equal to half the diagonal of the square flag to the right and down.

- What shapes were formed by the colored areas?

The black region was a square formed by the four congruent isosceles right triangles. The silver pieces were two congruent squares. The red pieces were four congruent trapezoids with one leg being an altitude of the trapezoid. The red pieces could have been cut and rearranged into a rectangle, but that did not make the problem easier to solve.

- What key ideas help you solve the problem?

We worked with half of the flag. We were able to arrange it into a square, so all we needed was the formula for the area of a square to help us get the side length of that square and the black square. The rest was basically computation.

- How could you extend this problem?

Change the shape of the flag to be a non-square rectangle. Change the design on the flag. Look for patterns in the different flags.

Student Work

A copy of a student's work on this task appears on the next page.

This work exemplifies many of the solution guide criteria. For example:

- Shows an understanding of the relationships among elements.

The student recognized the right triangles and used the Pythagorean Theorem. He recognized that one of the black right triangles is one-fourth of the black area of the square. He recognized that the segment labeled $x + y + z$ was the same as the side of the square—20 cm in length.

- Demonstrates geometric concepts, process, and skills.

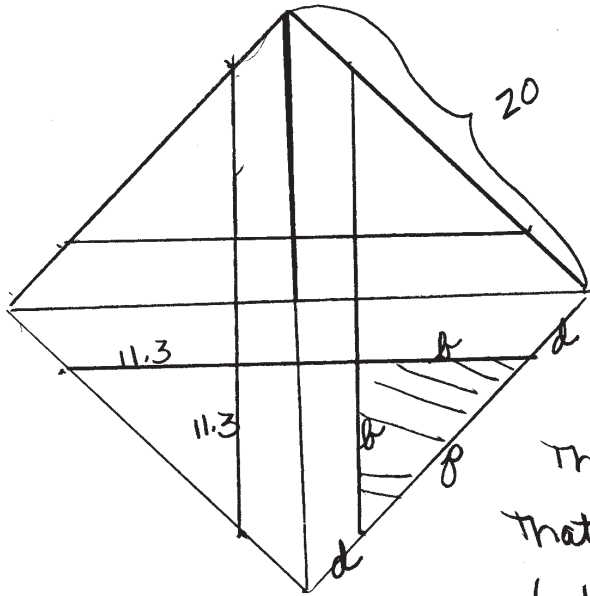
The student used the Pythagorean Theorem correctly. He showed the algebraic processes necessary to solve for the lengths of the unknown segments.

- Communicates clear, detailed, and organized solution strategy

Overall, the solution is organized clearly. However, when setting up equations, the student should have given a more detailed explanation of how and why the equations were set up the way they were. For example, the student set up $2x^2 = b^2$ but there is no explanation of why.

This student was advised that his solution would have been more complete if he had justified his steps and indicated the reasons for the steps in his solution. A teacher might ask the student some of the following questions to clarify his thinking:

- How did you know that the triangle with legs labeled b had two congruent legs?
- Why does $2b^2 = p^2$?
- How do you know that $20 - p = 2d$?
- What justifies your statement that $2x + y = 20$?



red 34%
silver 2%
black 64%

$$\frac{1}{4} \text{ of the square} = \frac{1}{4}(20)^2 = 100 \text{ in}^2$$

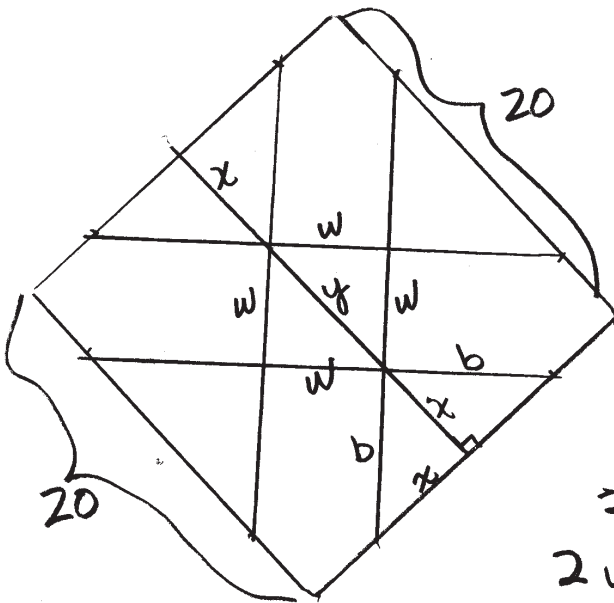
The black part is 64% of it or 64 in^2 .
That is one of the black triangles

$$\frac{1}{2}bh = 64 \quad b = h$$

$$\frac{1}{2}b^2 = 64 \quad b^2 = 128 \quad b \approx 11.3$$

$$2b^2 = p^2 \quad 2(128) = p^2 \quad p^2 = 256 \quad p = 16$$

$$20 - p = 2d \quad 20 - 16 = 2d \quad 4 = 2d \quad 2 = d$$



$$2x^2 = b^2 \quad 2x^2 = 128$$

$$x^2 = 64 \quad x = 8$$

$$2x + y = 20$$

$$2(8) + y = 20$$

$$16 + y = 20$$

$$y = 4$$

$$2w^2 = 4^2$$

$$w^2 = 8$$

$$2w^2 = y^2$$

$$2w^2 = 16$$

$$w = \sqrt{8}$$

$$w^2 = \text{area of silver} = 8 \quad \frac{8}{400} = 2\%$$

$$100\% - (2\% + 64\%) = \text{area of red} = 100\% - 66\% = 34\%$$

The Shortest Cable Line

Two houses are to be connected to cable TV by running cable lines from the houses to a common point of connection with the main cable. The main cable runs underground along the edge of the street the houses face. Without measuring, determine where that connection point should be placed to minimize the amount of cable run from the houses to the street. Explain your reasoning.





Notes

Materials:

One ruler per student
Unlined paper and patty paper, and geometry software

Geometry TEKS Focus:

(G.5) **Geometric patterns.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to:

(C) use properties of transformations and their compositions to make connections between mathematics and the real world, such as tessellations; and

(G.10) **Congruence and the geometry of size.** The student applies the concept of congruence to justify properties of figures and solve problems.

The student is expected to:

(A) use congruence transformations to make conjectures and justify properties of geometric figures including figures represented on a coordinate plane; and

Additional Geometry TEKS:

(G.4) **Geometric structure.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to select an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

Teacher Notes

Scaffolding Questions:

- What path gives the shortest distance between two points?
- How can you position three points to minimize the distance between them?
- What transformations can you use to create a straight line path between point B and an image of point A?

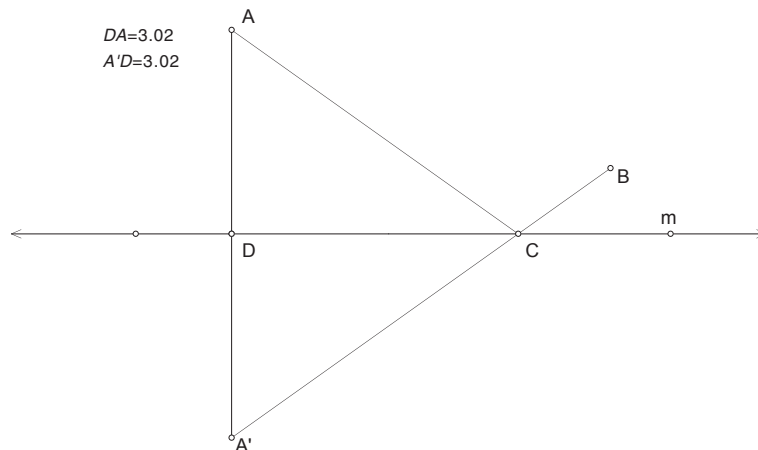
Sample Solutions:

Let C be the connection point along the street to which the cable lines from points A and B will run. To minimize the amount of cable run from the houses to this point, we need to minimize $AC + CB$. To do this we use congruence transformations.

Let line m represent the main cable.



Reflect point A over line m to get its image, A' . Draw $\overline{A'B}$ and label the point of intersection with line m as point C .



Since A' , C , and B are collinear points, $A'C + CB = A'B$ is the shortest distance between points A and B .

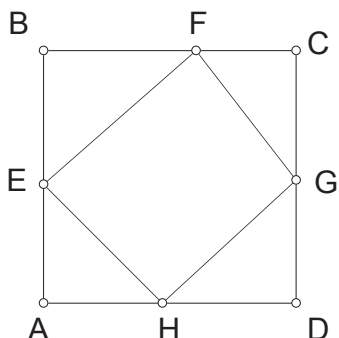
Reflecting point A over line m to get its image A' means that line m is the perpendicular bisector of $\overline{AA'}$, and any point on line m is equidistant from points A' and A . Therefore,

$$AC = A'C, \text{ and } A'B = AC + CB.$$

This shows that point C minimizes $AC + CB$.

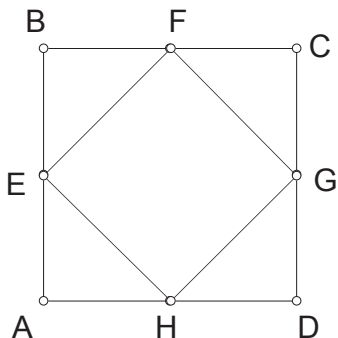
Extension Question:

- A square garden is to be divided up into four right triangular regions around a quadrilateral region with a pathway (EFGH) surrounding the quadrilateral region. One vertex of the quadrilateral is the midpoint of a side of the square garden, as shown below:



$ABCD$ is a square, and E is the midpoint of \overline{AB} . Where should the other vertices of quadrilateral $EFGH$ be to minimize its perimeter? Use congruence transformations (isometries) to investigate and explain your conclusion.

First, consider making quadrilateral $EFGH$ a square by placing F , G , and H at the midpoints of the other three sides.



Connections to TAKS:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

One student says . . .

In order to solve this problem, you should use the minimal path conjecture. First, reflect one of the points across the cable line. Then connect the image that results to the other point's pre-image. Once you've found the intersecting point, you connect each pre-image to it, and the result is the minimal path.

The student:

(A) uses congruence transformations to make conjectures and justify properties of geometric figures; and

(B) justifies and applies triangle congruence relationships.

Texas Assessment of Knowledge and Skills:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

II. Transformation: Reflections

Student Response:

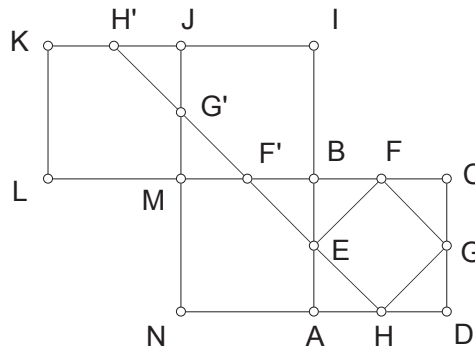
In order to solve this problem, you should use the minimal path conjecture. First, reflect one of the points across the cable line. Then connect the image that results to the other point's pre-image. Once you've found the intersecting point, you connect each pre-image to it, and the result is the minimal path.

Since EFGH is formed by connecting the midpoints of square ABCD, triangles EBF, FCG, GDH, and HAE are congruent isosceles right triangles.

This means that $\overline{EF} \cong \overline{FG} \cong \overline{GH} \cong \overline{HE}$, so that EFGH is a rhombus.

Since $m\angle EFB = m\angle CFG = 45^\circ$, $m\angle EFG = 180 - (45 + 45) = 90^\circ$, so that EFGH is a square.

We use reflections to show that the path, EF + FG + GH + EH, is equal in length to a straight line path, which is the shortest distance between two points.



Reflect square EFGH over line \overline{AB} to form $\overline{F'E} \cong \overline{FE}$. Reflect the newly positioned square, EFGH, over line \overline{BC} to form $\overline{F'G'} \cong \overline{FG}$. Finally, reflect this newly positioned square, EFGH, over line \overline{MJ} to form $\overline{G'H'} \cong \overline{GH}$.

Now H, E, F', G', and H' are collinear, and $\overline{HE} + \overline{EF'} + \overline{F'G'} + \overline{G'H'}$, which is the shortest distance between H and H', is equal to $\overline{HE} + \overline{EF} + \overline{FG} + \overline{GH}$, the perimeter of the inner square.

Chapter 7:
Similarity



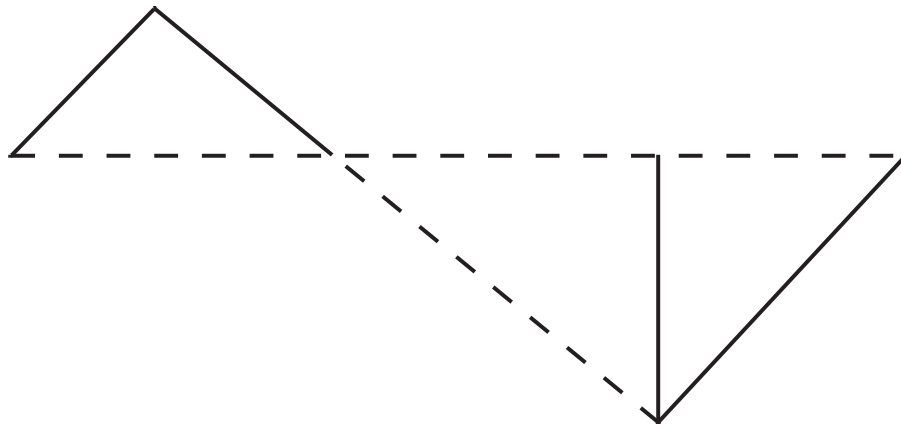
Introduction

In this chapter, the performance tasks require the students to apply the properties of similarity to justify properties of figures and to solve problems using these properties.

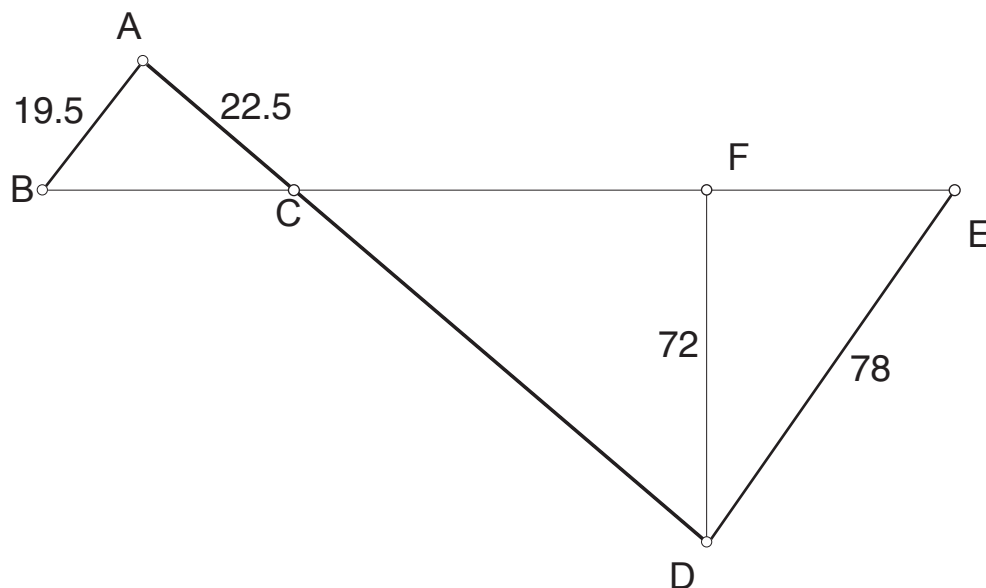


Ancient Ruins

Archaeologists flying over a remote area in the interior of Mexico saw what appeared to be the ruins of an ancient ceremonial temple complex below. This diagram shows what they saw:



Based on their photographs, they believed that some of the walls were still intact. These are drawn as solid segments, and the walls that had crumpled or had obviously been there are drawn as dashed segments. Before they can excavate the site, the archaeologists need to construct an accurate scale drawing or model. Based on their altitude flying over the ruins and the measurements made on the photograph, they generated the following drawing of the ruins:



The lengths, in feet, of the walls \overline{AB} , \overline{AC} , \overline{DE} , and \overline{DF} were 19.5, 22.5, 78, and 72, respectively. $\overline{AB} \parallel \overline{DE}$ and $\overline{DF} \perp \overline{CE}$.

To plan for the excavation, they need to know a number of things about the site.

1. The archaeologists estimate it will require 45 minutes to an hour to excavate each foot of the exterior walls of the temple site. Approximately how long will it take to complete this task? Explain in detail how you determined this.
2. The entrance into the ceremony preparation room appears to be along wall \overline{BC} and directly opposite vertex A . What would this mean geometrically? What is the distance from point A to the entrance? Were any of the walls of this room perpendicular to each other?
3. The archaeologists believe that about 9 square feet of space was needed in the main temple for each person. How many people could occupy the temple (triangle CDE)? How many priests and assistants could occupy what appears to be the ceremony preparation room (triangle ABC)?





Notes

Materials:

One calculator per student
Unlined paper and construction tools

Geometry TEKS Focus:

(G.11) **Similarity and the geometry of shape.** The student applies the concept of similarity to justify properties of figures and solve problems.

The student is expected to:

(A) use and extend similarity properties and transformations to explore and justify conjectures about geometric figures;

(B) use ratios to solve problems involving similar figures;

Additional Geometry TEKS:

(G.5) **Geometric patterns.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to:

(B) use numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles;

Teacher Notes

Scaffolding Questions:

- What information do you need to determine in order to compute the perimeters?
- What appears to be true about triangles ABC and CDE?
- What special segment on a triangle is question 3 referring to?
- Can the Pythagorean Theorem or Pythagorean Triples help you find answers to the questions?
- How are the ratios of corresponding linear dimensions on similar triangles related to the ratios of their areas? The ratio of their volumes?

Sample Solutions:

1. To find the perimeters the missing sides, segments BC, DC, and EC need to be found. Triangles ABC and DEC appear to be similar triangles.

Segments AB and DE are parallel, so $\angle ABC \cong \angle DEC$ because they are alternate interior angles on parallel lines. Because they are vertical angles, $\angle ACB \cong \angle DCE$.

Thus, $\triangle ABC \approx \triangle DEC$ because two angles of one triangle are congruent to two angles of the other triangle. Similar triangles have proportional corresponding sides:

$$\frac{AB}{DE} = \frac{BC}{EC} = \frac{AC}{DC} = \frac{1}{4} \text{ since } AB = 19.5 \text{ and } DE = 78.$$

$$\text{Use the proportion to get DC: } \frac{AC}{DC} = \frac{22.5}{DC} = \frac{1}{4} \text{ so } DC = 90.$$

To get EC, find FE and FC. Consider the right triangles DFE and DFC. Pythagorean Triples may be used.

Triangle DFE has $DE = 78 = 6$ times 13, and $DF = 72 = 6$ times 12, so $FE = 6$ times 5 = 30.

Triangle DFC has $DF = 72 = 18$ times 4, and $DC = 90 = 18$ times 5, so $FC = 18$ times 3 = 54.

Thus, $EC = 30 + 54 = 84$.

To get BC , use the proportion: $\frac{BC}{EC} = \frac{BC}{84} = \frac{1}{4}$, so $BC = 21$.

Perimeter of triangle $ABC = 19.5 + 22.5 + 21 = 63$ feet.

Perimeter of triangle $DEC = 78 + 84 + 90 = 252$ feet.

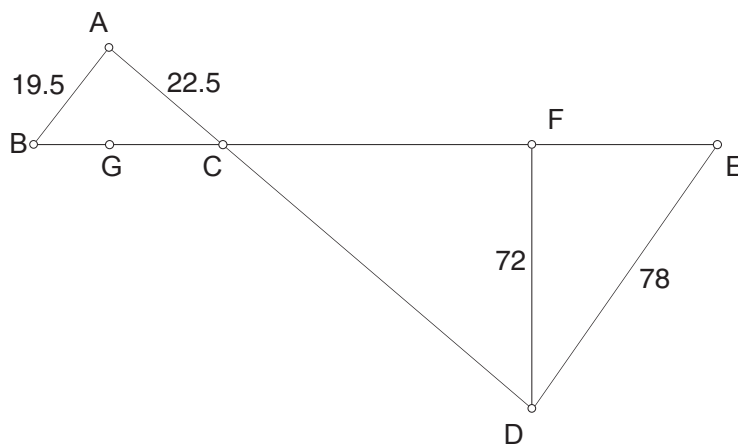
The time it will take to excavate the perimeters is

$(63 \text{ ft.} + 252 \text{ ft.}) \cdot (45 \text{ min. per foot}) = 14175 \text{ min.} = 236.25 \text{ hours}$

$(63 \text{ ft.} + 252 \text{ ft.}) \cdot (1 \text{ hour per foot}) = 315 \text{ hours}$

The excavation time will be between 236.25 hours and 315 hours.

- Since the entrance along wall \overline{BC} is directly opposite vertex A , the distance from point A to the entrance should be along the perpendicular from A to the line containing \overline{BC} . Let point G be the entrance. Then segment \overline{AG} is the altitude to segment \overline{BC} (since triangle ABC is acute).



To find the length of this altitude, use the fact that the ratio of corresponding altitudes on similar triangles is equal to the ratio of the lengths of the corresponding sides, so

$$\frac{AG}{DF} = \frac{AC}{DC} = \frac{1}{4}, \text{ and } AG = 18 \text{ feet.}$$

No pair of walls in this room is perpendicular to each other since $(19.5)^2 + (21)^2 \neq (22.5)^2$

- Since $\overline{DF} \perp \overline{EC}$, \overline{DF} is the altitude to \overline{EC} .

Thus, the area of $\triangle DEC = \frac{1}{2}(72)(84) = 3024$ square feet

(G.11) **Similarity and the geometry of shape.** The student applies the concept of similarity to justify properties of figures and solve problems.

The student is expected to:

(C) develop, apply, and justify triangle similarity relationships, such as right triangle ratios, trigonometric ratios, and Pythagorean triples using a variety of methods; and

(D) describe the effect on perimeter, area, and volume when one or more dimensions of a figure are changed and apply this idea in solving problems.

Connections to TAKS:

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Dividing the area by 9 square feet of space per person shows that the temple could have housed 336 people.

To get the area of $\triangle ABC$, either use the triangle area formula or the property that the ratio of the areas of similar triangles is the square of the ratio of the corresponding sides:

$$\frac{\text{area of } \triangle ABC}{\text{area of } \triangle DEC} = \frac{A}{3024} = \left(\frac{1}{4}\right)^2$$

Solving this for A shows the area to be 189 square feet. Dividing this by 9 square feet reveals that the ceremony preparation room could have housed 21 priests and assistants.

Extension Questions:

- In the proposal that the archaeologists must write to excavate the site, they must compare the perimeters and the areas of the ceremony preparation room to those of the main temple. How could this be done?

The ratio of the areas is the square of the ratio of the corresponding sides of the similar figures. Here is why:

On the first triangle let S_1 and h_1 be a side and the altitude to that side, respectively.

On the second triangle let S_2 and h_2 be the corresponding side and the altitude to those in the first triangle, respectively. Then $\frac{S_2}{S_1} = \frac{h_2}{h_1} = r$ so that $S_2 = rS_1$ and $h_2 = rh_1$.

$$\text{Now } \frac{\text{area of triangle 2}}{\text{area of triangle 1}} = \frac{\frac{1}{2}h_2S_2}{\frac{1}{2}h_1S_1} = \frac{(rh_1)(rS_1)}{h_1S_1} = r^2,$$

which is the square of the ratio of the sides.

- It is believed that each of the two rooms were built in the shape of a triangular pyramid. What can be said about the capacity of the rooms? Is it possible to determine their capacity?

If it is assumed that the triangular pyramids were similar figures, then the ratio of their volumes is the cube of the ratio of corresponding sides.

In this case, the ratio of the volumes would be $\left(\frac{1}{4}\right)^3 = \frac{1}{64}$.

To compute the volumes, compute the following:

$$\frac{1}{3}(\text{altitude to pyramid base})(\text{base area}).$$

Even though the ratio of the volumes is known, there is not enough information to find the volumes. Both base areas can be computed. The altitude of one of the triangular pyramids must be known to find the altitude of the other. If one of the altitudes were known, the volumes could be computed.

Student Work Sample

A student working individually did the work sample on the next page.

- Demonstrates geometric concepts, processes and skills

The student applied a ratio for similarity and the Pythagorean Theorem to find the length of the missing sides in the problem. He also found correctly corresponding sides to set up equations for the ratio.

Although the way the solution is organized is clear in both symbolic and verbal forms, it is still missing an explanation of why the student could apply certain mathematical principles in the situation. For example, the student used a ratio to find the length of \overline{CD} without explaining why he could use a ratio. He should have explained that the triangles ABC and CDE are similar so that he can use a ratio to find the missing length.

Ancient Ruins

$$1) \frac{19.5}{225} = \frac{78}{x}$$

$$19.5x = 1755$$

$$x = 90 = CD$$

$$90^2 - 72^2 = CF^2$$

$$8100 - 5184 = CF^2$$

$$\sqrt{2916} = 54 = CF$$

$$78^2 - 72^2 = FE^2$$

$$6084 - 5184 = FE^2$$

$$\sqrt{900} = 30 = FE$$

$$\frac{19.5}{x} = \frac{78}{84}$$

$$78x = 1638$$

$$x = 21 = BC$$

time = 236 hrs 15 min

first we setup a ratio of $\frac{AB}{AC} = \frac{ED}{CD}$ to find CD. Then we used pythagorean theorem to find CF + FE to get CE. Next we set up the proportion $\frac{BC}{CE}$ to get 21. Finally we found the sum of all sides that multiplied it by 215 min then divided it by 60 and converted it into time.

2) That $AX \perp BC$

$$\frac{19.5}{x} = \frac{78}{72}$$

$$78x = 1404$$

$$x = 18 = AX \perp BC$$

$$3) A = \frac{B \times H}{2} = \Delta CDE$$

$$A = 3024 = \Delta CDE$$

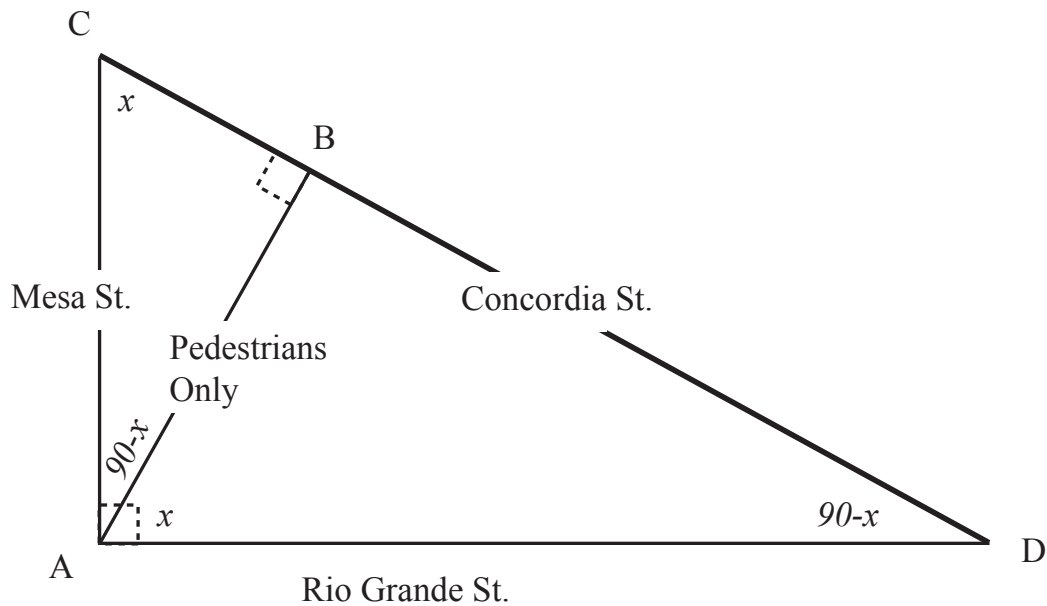
336 ft² per person in ΔCDE

$$A = \frac{b \times h}{2} = \Delta ABC$$

$$A = \frac{18 \times 21}{2} = \frac{378 \text{ ft}^2}{2} \quad 21 \text{ ft}^2 \text{ per person in } \Delta ABC$$



Sightseeing Walk



Yuma City has an historic region downtown between Mesa Street, Rio Grande Street, and Concordia Street. Mesa and Rio Grande intersect to form a right angle. There is a six-block Pedestrians Only path from the intersection of Mesa and Rio Grande to Concordia that intersects Concordia at a right angle. A sightseer started at the intersection of Mesa and Rio Grande and walked along the Pedestrians Only path to Concordia. She then walked four blocks along Concordia to Mesa and then to the intersection of Mesa and Rio Grande. After that she walked along Rio Grande to the intersection of Rio Grande and Concordia.

Answer the following questions, completely justifying your answers with geometric explanations.

1. How far did the sightseer walk?
2. If another sightseer had started at the intersection of Mesa and Concordia and walked along Concordia Street to the intersection of Concordia and Rio Grande, how far would he have walked?



Notes

Materials:

One ruler and compass per student

Unlined paper, patty paper, and geometry software

Geometry TEKS Focus:

(G.11) **Similarity and the geometry of shape.** The student applies the concept of similarity to justify properties of figures and solve problems.

The student is expected to:

(A) use and extend similarity properties and transformations to explore and justify conjectures about geometric figures;

(B) use ratios to solve problems involving similar figures;

Additional Geometry TEKS:

(G.5) **Geometric patterns.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to:

(B) use numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles;

(G.8) **Congruence and the geometry of size.** The student uses tools to determine measurements of geometric figures and extends measurement concepts to find perimeter, area, and volume in problem situations.

Teacher Notes

Scaffolding Questions:

- What segments represent the paths taken by the sightseer?
- What are the known distances?
- What are the unknown distances?
- What types of triangles do you see in this problem?
- What relationships about this/these triangle types can help you find unknown quantities?

Sample Solutions:

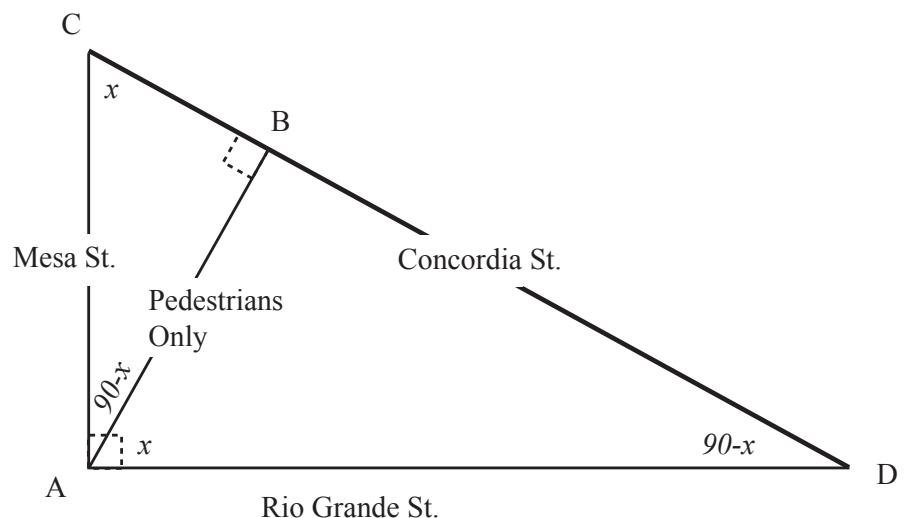
1. The distance the sightseer walks is given by $AB + BC + CA + AD$.

We know that $AB = 6$ and $BC = 4$. To find CA we apply the Pythagorean Theorem to right triangle ABC .

$$CA = \sqrt{AB^2 + BC^2} = \sqrt{36 + 16} = \sqrt{52} \approx 7.21 \text{ blocks}$$

There are three right triangles: $\triangle CAD$, $\triangle CBA$, and $\triangle ABD$, with right angles $\angle CAD$, $\angle CBA$, and $\angle ABD$.

In the large right triangle CAD , let the measure of acute angle C be x degrees. Then, since the acute angles of a right triangle are complementary, the measure of angle D is $90 - x$ degrees.



In triangle CBA, the measure of angle C is x degrees, so the measure of angle CAB is $90 - x$ degrees.

In triangle ABD, the measure of angle D is $90 - x$ degrees, so the measure of angle BAD is x degrees. Now we have $\triangle CAD \sim \triangle CBA \sim \triangle ABD$, because if three angles of one triangle are congruent to the corresponding three angles of another triangle, the triangles are similar. (Angle-Angle-Angle similarity)

Using triangles CBA and ABD, we have

$$\frac{CB}{AB} = \frac{CA}{AD}$$

$$\frac{4}{6} = \frac{\sqrt{52}}{AD}$$

$$AD = \frac{6\sqrt{52}}{4}$$

$$AD \approx 10.82 \text{ blocks}$$

The sightseer walked $AB + BC + CA + AD = 6 + 4 + 7.21 + 10.82 = 28.03 \approx 28$ blocks.

2. The distance from the intersection of Mesa and Concordia to the intersection of Concordia and Rio Grande is given by $CD = CB + BD = 4 + BD$.

To find BD we can use the same similar triangles as in Problem 1:

$$\frac{CB}{AB} = \frac{AB}{BD}$$

$$\frac{4}{6} = \frac{6}{BD}$$

$$BD = 9 \text{ blocks}$$

The distance from Mesa and Concordia to Concordia and Rio Grande is 4 plus 9, or 13 blocks.

The student is expected to:

(C) derive, extend, and use the Pythagorean Theorem; and

(G.11) **Similarity and the geometry of shape.** The student applies the concept of similarity to justify properties of figures and solve problems.

The student is expected to:

(C) develop, apply, and justify triangle similarity relationships, such as right triangle ratios, trigonometric ratios, and Pythagorean triples using a variety of methods; and

Connections to TAKS:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

One teacher says . . .

“It surprised me how hard they worked on this problem, because they are usually difficult to keep on task. I was pleased with how quickly they recognized similar triangles and how well they set up the proportions because the concept was still new to them. I was disappointed in their written explanations because they were usually incomplete although they understood the problem. The instructional strategies I will modify to improve student success is that I will insist on better-written explanations for all problems.”

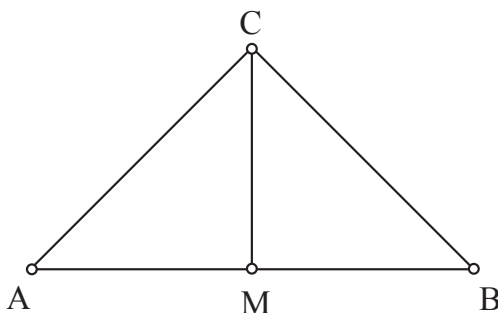
Extension Questions:

- What important geometric concepts did you consider in this problem?

To find the missing distances along the sightseer's route, we showed we had three similar right triangles. Using the resulting proportions, we were able to find all missing dimensions along the streets.

- On the other side of Concordia Street is an old cemetery in which a famous sheriff, Mo Robbins, is believed to be buried. The cemetery forms an isosceles right triangle with legs 10 blocks long. The hypotenuse of the triangular cemetery runs along Concordia Street. Sheriff Robbins' gravesite is believed to be located at the centroid of the triangular cemetery, but there is no tombstone to mark his grave. How would you locate the gravesite of this famous person?

Cemetery:



The centroid is the intersection of the medians. Its distance from each vertex is two-thirds the length of that median. Since triangle ACB is an isosceles right triangle with right angle C , the median from vertex C is the median to the hypotenuse. Triangle CMB is also a right isosceles triangle. $CM = MB$. Therefore, the median CM is half the length of the hypotenuse, \overline{AB} .

Because the legs of isosceles right triangle ACB are 10 blocks long,

$$AB = 10\sqrt{2} \text{ and } CM = \frac{1}{2}AB = 5\sqrt{2}.$$

The centroid is located $\frac{2}{3}(5\sqrt{2}) = 4.71$ blocks along segment CM from point C .

- Suppose the triangular region formed by the streets were an equilateral triangle and the distance between any two intersections were 10 blocks. How will your answers to Problem 1 change?

Since the triangular region formed by the streets is equilateral, the pathway, segment \overline{AB} , is an altitude and divides the region into two 30-60-90 triangles. This is because an altitude on an equilateral triangle is also an angle bisector and a median.

In 30-60-90 triangle ACB , the hypotenuse $AC = 10$, so the shorter leg $CB = 5$ and the longer leg $AB = 5\sqrt{3} \approx 8.66$ blocks. Also $AC = DA$.

The sightseer walked $AB + BC + AC + AD = 8.66 + 5 + 10 + 10 = 33.66$ blocks.



Student Work Sample

An individual student produced the work on the next page. The work exemplifies the following criterion:

- Uses geometric and other mathematical principles to justify the reasoning used to analyze the problem.

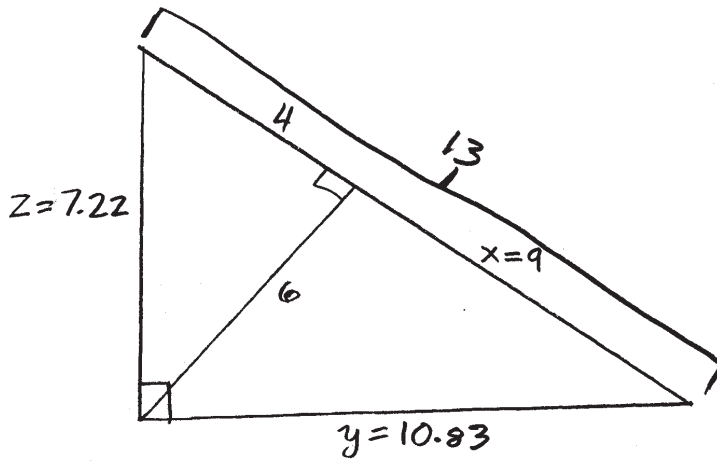
The student applied theorems and corollaries correctly to get answers in that she substituted the correct numbers in the equations provided by corollaries. The student referenced specific corollaries from the book, but she did not write what they meant. The student also provided a good combination of verbal explanation and symbolic process.

The following criterion that was not met:

- Uses appropriate terminology and notation.

The student inappropriately used equality signs and used x to represent two different things (one for the altitude and the other for the missing part of hypotenuse).

Sight Seeing Walk



① I used corollary theorem 10.2, which states, if the altitude is drawn to the hypotenuse of the right triangle, then its (altitude) length is the geometric mean of the length of two segments of the hypotenuse. $\frac{\text{one side}}{x} = \frac{x}{\text{2nd side}}$, this allowed me to swap it and find x (one of the segments of the hypotenuse). $\frac{6}{z} = \frac{z}{x} = 36 = 4x = \boxed{9=x}$

I also used the pythagorean theorem, which states if a triangle is a right triangle with legs of length $A+B$ and hypotenuse of length C , then $a^2 + b^2 = c^2$

$$6^2 + 4^2 = c^2$$

$$36 + 16 = c^2$$

$$\sqrt{52} = c$$

$$7.22 = c$$

$$\boxed{z = 7.22}$$

I used corollary 10.3, which states, the leg of a right triangle is the geometric mean of the total and adjacent part of the hypotenuse

$$\frac{\text{leg}}{\text{total hyp}} = \frac{\text{leg}}{\text{adjacent segment of hypotenuse}}$$

$$\frac{13}{y} = \frac{4}{9} = \frac{y^2}{117}$$

$$\sqrt{y^2} = \sqrt{117}$$

$$y = 10.83$$

I added the lengths of Mesa St. + Concord Street + Pedestrian + Rio Grande to get the distance she walked = 28.05 blocks

② I added the lengths of the two segments of Concordia St. that are divided by Pedestrian to get the total length of Concordia St.

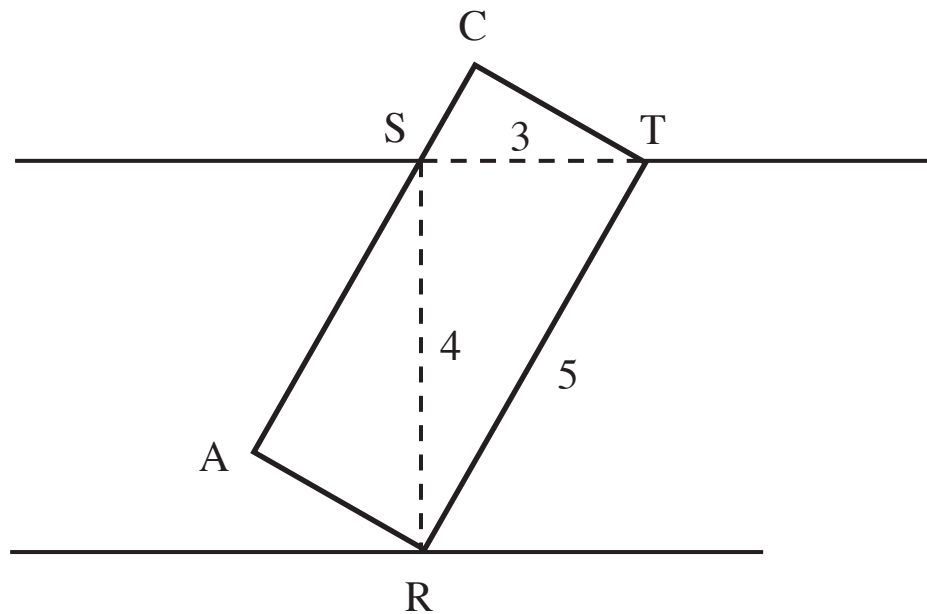
$$\begin{array}{r} 9 \\ +4 \\ \hline 13 \end{array}$$

$$\boxed{13 \text{ blocks}}$$



Will It Fit?

Rectangular cartons that are 5 feet long need to be placed in a storeroom that is located at the end of a hallway. The walls of the hallway are parallel. The door into the hallway is 3 feet wide and the width of the hallway is 4 feet. The cartons must be carried face up. They may not be tilted. Investigate the width and carton top area that will fit through the doorway.





Notes

Materials:

One calculator per student

Geometry TEKS Focus:

(G.11) **Similarity and the geometry of shape.** The student applies the concept of similarity to justify properties of figures and solve problems.

The student is expected to:

(B) use ratios to solve problems involving similar figures;

(C) develop, apply, and justify triangle similarity relationships, such as right triangle ratios, trigonometric ratios, and Pythagorean triples using a variety of methods; and

Additional Geometry TEKS:

(G.5) **Geometric patterns.** The student uses a variety of representations to describe geometric relationships and solve problems.

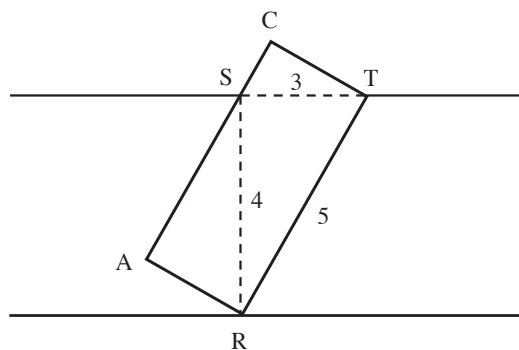
The student is expected to:

(D) identify and apply patterns from right triangles to solve meaningful problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

Teacher Notes

Scaffolding Questions:

- What segments represent the width of the carton?
- How does knowing the dimensions of triangle TSR help you?
- What triangles involve the width of the carton?
- What triangles involve the width of the hallway?
- Explain any angle relationships in the triangles.
- How are the triangles you see related? How does this help you?
- How can you use your solution to Problem 1 to help you with Problem 2?

Sample Solutions:

Triangle SRT is a right triangle because it has three sides that are Pythagorean Triples. That is, the sum of the squares of two sides is the square of the third side: $3^2 + 4^2 = 5^2$.

Triangle SCT is a right triangle because $\overline{AC} \perp \overline{CT}$.

$\overline{AC} \parallel \overline{RT}$ so the alternate interior angles are congruent.

$\angle RSA \cong \angle TRS$.

$\angle TRS$ and $\angle STR$ are complementary angles because they are the two acute angles of a right triangle SRT.

$\angle CTS$ and $\angle STR$ are complementary angles because the figure CTRA is a rectangle with right angle CTR.

$\angle CTS \cong \angle TRS$ because they are two angles that are

complementary to the same angle.

Thus, the three right triangles, $\triangle CTS$, $\triangle SRT$, and $\triangle ASR$, have congruent acute angles: $\angle CTS \cong \angle TRS \cong \angle RSA$.

Therefore, $\triangle CTS \approx \triangle SRT \approx \triangle ASR$ because they are right triangles with acute angles that are congruent.

Now write the proportion between the corresponding sides:

$$\frac{CT}{SR} = \frac{ST}{TR}$$

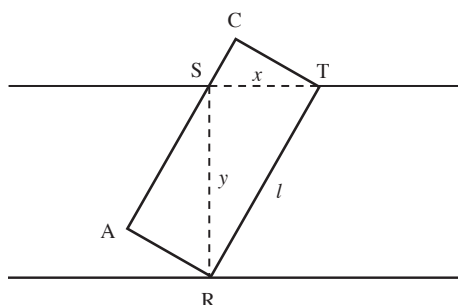
$$\frac{CT}{4} = \frac{3}{5}$$

$$CT = 2.4$$

The widest carton that will fit through the opening has a width of 2.4 feet, and the top surface area is 2.4 feet times 5 feet = 12 square feet.

Extension Question:

- Generalize your results for Problem 1 for a hallway opening of x feet and a hallway width of y feet if the maximum carton dimensions are l feet length and $x^2 + y^2 = l^2$.



Because $x^2 + y^2 = l^2$, the triangle STR is a right triangle.

$$\frac{CT}{SR} = \frac{ST}{TR}$$

Let $CT = w$.

$$\frac{w}{y} = \frac{x}{l}$$

$$w = \frac{xy}{l}$$

(G.8) **Congruence and the geometry of size.** The student uses tools to determine measurements of geometric figures and extends measurement concepts to find perimeter, area, and volume in problem situations.

The student is expected to:

(C) derive, extend, and use the Pythagorean Theorem; and

Connections to TAKS:

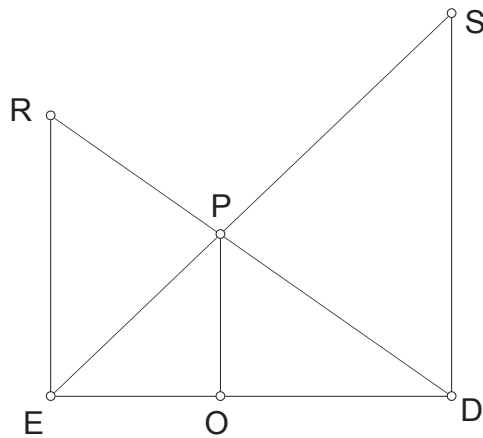
Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

This shows that the maximum carton width is the product of the door width and the hallway width divided by the carton length. The maximum carton surface is the width, w , times the length, l , or $w \cdot l = \frac{xy}{l} \cdot l = xy$. The maximum carton area is the product of the door width and the hallway width.

Spotlights

1. The spotlight display for an outdoor rock concert is being planned. At the sides of the stage a red spotlight is mounted on a pole 18 feet high, and a green spotlight is mounted on a pole 27 feet high. The light from each spotlight must hit the base of the other pole as shown in the diagram. How high above the ground should the stage be so that the spotlights meet and highlight the upper body of a performer who is about 6 feet tall?

The red spotlight is at point R . The green spotlight is at point S . Ground level is segment \overline{ED} .



2. If the poles are 30 feet apart and the stage is 20 feet long, how should the stage be positioned so that the spotlights meet on the upper body of the performer when he is center stage?
3. Generalize your results in Problem 1 if the poles for the spotlights are a feet and b feet long.



Notes

Materials:

One ruler per student

Unlined paper and geometry software

Geometry TEKS Focus:

(G.11) **Similarity and the geometry of shape.** The student applies the concept of similarity to justify properties of figures and solve problems.

The student is expected to:

(A) use and extend similarity properties and transformations to explore and justify conjectures about geometric figures;

(B) uses ratios to solve problems involving similar figures;

Additional Geometry TEKS:

(G.7) **Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student is expected to:

(A) use one- and two-dimensional coordinate systems to represent points, lines, rays, line segments, and figures;

(B) use slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons; and

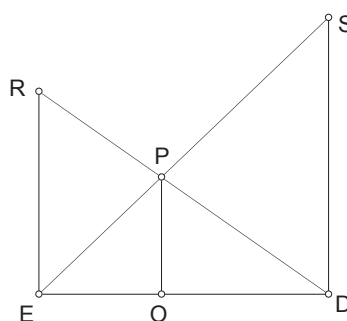
Teacher Notes

Scaffolding Questions:

- In what figures do you see segment \overline{PO} as a side or special segment?
- Since \overline{PO} is perpendicular to segment \overline{ED} , what does this tell you about \overline{PO} and triangle EPD ? Does this help? Why or why not?
- Can you find similar triangles involving segment \overline{PO} ?
- What proportions can you write involving \overline{PO} ?
- How can you use algebraic notation to make your explanation easier to follow?
- Experiment with equivalent ways of writing a proportion to discover a way to solve for the length of \overline{PO} .
- Does the distance between the light poles seem to matter?
- How can you generalize from poles of 18 feet and 27 feet to poles of arbitrary lengths, a feet and b feet?

Sample Solutions:

1.



Let $h = PO$, which is the height at which the spotlight beams meet.

We know that $RE = 18$ and $SD = 27$. We can use two pairs of similar triangles to compare h to the lengths of the poles.

$\triangle EOP \sim \triangle EDS$ by Angle-Angle-Angle similarity. The triangles have right angles $\angle EOP$ and $\angle EDS$, and they share the common angle, $\angle PEO$.

Similarly, $\triangle DOP \sim \triangle DER$.

Let $EO = x$ and $OD = y$. Consider the following proportions that we get from the similar triangles:

$$1. \quad \frac{PO}{SD} = \frac{EO}{ED} \Rightarrow \frac{h}{27} = \frac{x}{x+y}$$

$$2. \quad \frac{PO}{RE} = \frac{OD}{ED} \Rightarrow \frac{h}{18} = \frac{y}{x+y}$$

Experimenting with properties of proportions gives an equation with h and the pole heights:

$$1. \quad \frac{x}{h} = \frac{x+y}{27}$$

$$2. \quad \frac{y}{h} = \frac{x+y}{18}$$

Adding these equations results in the equation

$$\begin{aligned} \frac{x}{h} + \frac{y}{h} &= \frac{x+y}{27} + \frac{x+y}{18} \\ \frac{x+y}{h} &= \frac{x+y}{27} + \frac{x+y}{18} \\ (x+y)\frac{1}{h} &= (x+y)\left(\frac{1}{27} + \frac{1}{18}\right) \\ \frac{1}{h} &= \frac{1}{27} + \frac{1}{18} = \frac{18+27}{(27)(18)} = \frac{45}{(27)(18)} \end{aligned}$$

Take the reciprocal of both sides.

$$h = \frac{(27)(18)}{45} = \frac{54}{5} = 10.8 \text{ feet}$$

This shows that the spotlight beams will meet at a point 10.8 feet above the ground.

Therefore, if the performer is about 6 feet tall, the stage should be 4.8 feet off the ground.

(G.11) **Similarity and the geometry of shape.** The student applies the concepts of similarity to justify properties of figures and solve problems.

The student is expected to:

(C) develop, apply, and justify triangle similarity relationships, such as right triangle ratios, trigonometric ratios, and Pythagorean triples using a variety of methods; and

Connections to TAKS:

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

It also shows that the distance between the poles, $x + y$, does not matter. This distance was not needed to determine $h = 10.8$ feet.

2. The spotlight poles are to be 30 feet apart, so $x + y = 30$. The point at which the spotlights meet must be determined. In other words, find the value of x or y .

$$\begin{aligned}\frac{x}{x+y} &= \frac{h}{27} \\ \frac{x}{30} &= \frac{10.8}{27} \\ x &= 10.8 \cdot \frac{30}{27} \\ x &= 12 \text{ feet}\end{aligned}$$

The spotlights meet 12 feet in from the red spotlight and, therefore, 18 feet in from the green spotlight. We want this point to correspond to the center of the 20-foot-long stage.

We need 10 feet of the stage to the left of this point and 10 feet of this stage to the right of this point. This means the pole for the red spotlight should be 2 feet from the left edge of the stage, and the pole for the green spotlight should be 8 feet from the right edge of the stage.

3. To generalize our results for spotlight poles of lengths a feet and b feet, we would have the same pairs of similar triangles and the same proportions. We simply need to replace 18 feet with a feet and 27 feet with b feet.

$$\begin{aligned}\frac{1}{h} &= \frac{1}{b} + \frac{1}{a} \\ \frac{1}{h} &= \frac{a+b}{ab} \\ h &= \frac{ab}{a+b}\end{aligned}$$

Extension Questions:

- What are the key concepts needed to solve Problem 1?

We needed to relate the height at which the spotlight beams meet to the lengths of the poles. Segment \overline{PO} is the altitude from point P to \overline{ED} in triangle EPD. This does not help since we know nothing about this triangle. By the same reasoning, it does not help to consider \overline{PO} as a leg of right triangles EOP or POD.

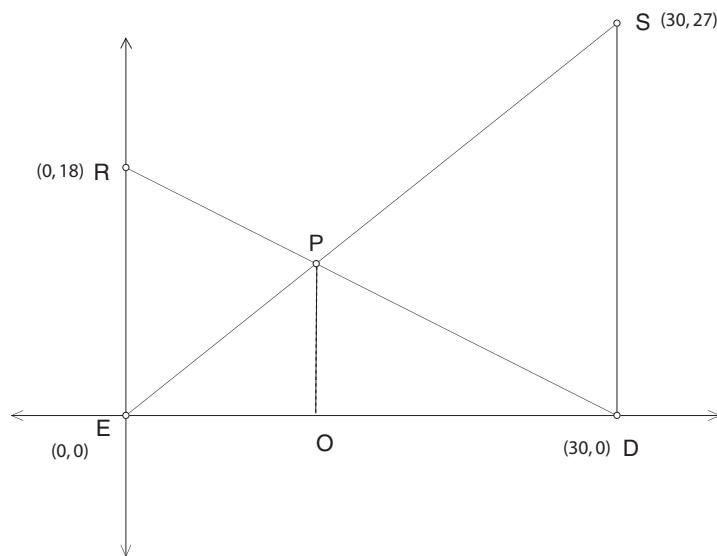
The big idea to use is similar triangles so we can write proportions involving \overline{PO} and the lengths of the light poles.

- Why doesn't the distance between the poles matter?

Segment \overline{PO} divides segment \overline{ED} into two pieces: segments \overline{EO} and \overline{OD} . When we solve the proportions for \overline{PO} , the sum of these unknown lengths is a factor on both sides of the equation.

- How could you use algebra and a coordinate representation to solve this problem?

The coordinates of the base of the stage can be (0,0) and (30,0). The red spotlight will be located at (0,18), and the green spotlight will be located at (30,27). The diagram below shows this.



The slope of segment \overline{RD} is $\frac{-18}{30} = \frac{-3}{5}$, and its y-intercept is 18.

The slope of segment \overline{ES} is $\frac{27}{30} = \frac{9}{10}$, and its y-intercept is 0.

The equations of these two segments are $y = \frac{-3}{5}x + 18$ and $y = \frac{9}{10}x$.

Solve this system to get $x = 12$. Substitute for x in either equation and solve to get $y = 10.8$.



References

Balanced Assessment Project Team. (2000). *Balanced Assessment for the Mathematics Curriculum: High School Assessment, Package 1*. White Plains, NY: Dale Seymour Publications.

Balanced Assessment Project Team. (2000). *Balanced Assessment for the Mathematics Curriculum: High School Assessment, Package 2*. White Plains, NY: Dale Seymour Publications.

National Council of Teachers of Mathematics. (2000). *Mathematics Assessment: Cases and Discussion Questions for Grades 6–12*. Classroom Assessment for School Mathematics Series. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (1995). *Assessment Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

University of St. Andrews, St. Andrews Scotland. (1996, December). Retrieved August 8, 2002 from <http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Steiner.html>.

