

# Geometry Assessments

*performance tasks aligned to the Texas standards*



**The Charles A. Dana Center**  
at The University of Texas at Austin

A publication of the Charles A. Dana Center  
an organized research unit of  
The University of Texas at Austin

**2007**



## **About The University of Texas at Austin Charles A. Dana Center**

The Charles A. Dana Center supports education leaders and policymakers in strengthening education. As a research unit of The University of Texas at Austin’s College of Natural Sciences, the Dana Center maintains a special emphasis on mathematics and science education. The Dana Center’s mission is to strengthen the mathematics and science preparation and achievement of all students through supporting alignment of all the key components of mathematics and science education, prekindergarten–16: the state standards, accountability system, assessment, and teacher preparation. We focus our efforts on providing resources to help local communities meet the demands of the education system—by working with leaders, teachers, and students through our Instructional Support System; by strengthening mathematics and science professional development; and by publishing and disseminating mathematics and science education resources.

### **About the development of this book**

The Charles A. Dana Center has developed this standards-aligned mathematics education resource for mathematics teachers.

The development and production of the first edition of Geometry Assessments was supported by the National Science Foundation and the Charles A. Dana Center at the University of Texas at Austin. The development and production of this second edition was supported by the Charles A. Dana Center. Any opinions, findings, conclusions, or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation or The University of Texas at Austin.

The second edition updates the Texas Essential Knowledge and Skills statements and alignment charts to align with the state’s 2005–06 revisions to the Mathematics TEKS. The assessments have also been re-edited for clarity and to correct some minor errors.

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## TEKS and TAKS Resources

The mathematics Texas Essential Knowledge and Skills (TEKS) were developed by the state of Texas to clarify what all students should know and be able to do in mathematics in kindergarten through grade 12. Districts are required to provide instruction that is aligned with the mathematics TEKS, which were originally adopted by the State Board of Education in 1997 and implemented statewide in 1998. Revisions to the Mathematics TEKS were adopted in 2005–06 and implemented starting in Fall 2006. The mathematics TEKS also form the objectives and student expectations for the mathematics portion of the Texas Assessment of Knowledge and Skills (TAKS).

The mathematics TEKS can be downloaded in printable format, free of charge, from the Dana Center’s Mathematics TEKS Toolkit website ([www.mathtekstoolkit.org](http://www.mathtekstoolkit.org)). Perfect-bound and spiral-bound versions of the mathematics and science TEKS booklets are available for a fee (to cover the costs of production) from the Charles A. Dana Center at The University of Texas at Austin ([www.utdanacenter.org](http://www.utdanacenter.org)).

Resources for implementing the mathematics TEKS are available through the Charles A. Dana Center, regional education service centers, and the Texas Education Agency. Online resources can be found at in the Dana Center’s Mathematics TEKS Toolkit at [www.mathtekstoolkit.org](http://www.mathtekstoolkit.org).

The following products and services are also available from the Dana Center at [www.utdanacenter.org/catalog](http://www.utdanacenter.org/catalog):

- Revised Mathematics TEKS booklets and Mathematics TEKS charts for K–8 and 6–12
- *Mathematics Standards in the Classroom: Resources for Grades 3–5 and 6–8*
- *Middle School Mathematics Assessments: Proportional Reasoning*
- *Algebra I Assessments*
- *Algebra II Assessments*
- Professional development for mathematics and science teachers
- Professional development modules for educators, including administrators, teachers, and other leaders





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## Introduction:

### The importance of mathematics assessment

The Dana Center developed *Geometry Assessments* as a resource for teachers to use to provide ongoing assessment integrated with Geometry mathematics instruction.

The National Council of Teachers of Mathematics (2000) lists as one of its six principles for school mathematics that “Assessment should support the learning of important mathematics and furnish useful information to both teachers and students.”<sup>1</sup>

Further, NCTM (1995)<sup>2</sup> identified the following six standards to guide classroom assessment:

The Mathematics Standard:	Assessment should reflect the mathematics that all students need to know and be able to do.
The Learning Standard:	Assessment should enhance mathematics learning.
The Equity Standard:	Assessment should promote equity.
The Openness Standard:	Assessment should be an open process.
The Inferences Standard:	Assessment should promote valid inferences about mathematics learning.
The Coherence Standard:	Assessment should be a coherent process.

### What are the Geometry Assessments?

Teachers can use these *Geometry Assessments* to provide ongoing assessment integrated with Geometry instruction. The performance tasks, which embody what all students need to know and be able to do in a high school Geometry course, may be used for formative, summative, or ongoing assessment. The tasks are designed to diagnose students’ understanding of concepts and their procedural knowledge, rather than simply determine whether they reached the “right” or “wrong” answers. Teachers should assess frequently to monitor individual performance and guide instruction.

The purpose of these assessments is to

- clarify for teachers, students, and parents what is being taught and learned in Geometry,
- help teachers gain evidence of student insight and misconceptions to use as a basis for instructional decisions, and
- provide teachers with questioning strategies to guide instruction and enhance student learning.

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<sup>1</sup> National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics. (Summary available at [standards.nctm.org](http://standards.nctm.org).)

<sup>2</sup> National Council of Teachers of Mathematics. (1995). *Assessment Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics, pages 11, 13, 15, 17, 19, and 21.

## What's new?

The first edition of *Geometry Assessments* was published in 2002. In this second edition, we have included the 2005 revised secondary mathematics Texas Essential Knowledge and Skills and updated the TEKS alignment charts.

## How do the assessments support TEKS-based instruction?

Each performance task in the assessments book

- is aligned with the revised Geometry TEKS student expectations, and
- is aligned with the grade 11 exit-level Texas Assessment of Knowledge and Skills (TAKS) objectives.

## How are the assessments structured?

Teachers may use these assessments formatively or summatively, for individual students or groups of students. Each assessment

- includes a performance task,
- is aligned with the Geometry mathematics TEKS knowledge and skills as well as with student expectations,
- is aligned with the TAKS objectives,
- includes “scaffolding” questions that the teacher may use to help the student analyze the problem,
- provides a sample solution,\* and
- includes extension questions to bring out additional mathematical concepts in a summative discussion of solutions to the problem.

\*The sample solution is only one way that a performance task may be approached and is not necessarily the “best” solution. For many of the tasks, there are other approaches that will also provide a correct analysis of the task. The authors have attempted to illustrate a variety of mathematical approaches in the various sample solutions. Several of the assessments also include samples of anonymous student work.

*Geometry Assessments* presents performance tasks based on the five strands in the Geometry TEKS—we have subdivided the tasks into seven categories:

1. Coordinate Geometry
2. Patterns, Conjecture, and Proof
3. Properties and Relationships of Geometric Figures
4. Area, Perimeter, and Volume
5. Solids and Nets
6. Congruence
7. Similarity

### **What is the solution guide?**

The solution guide is a one-page problem-solving checklist in the front of the book that teachers may use to track what is necessary for a student to give a complete solution for a given performance task. Because students need to know what criteria are expected in their solution, when assigning the performance task, the teacher can give students copies of a solution guide customized with marks indicating which of the criteria should be considered in the performance task analysis. For most performance tasks, all the criteria will be important, but initially the teacher may want to focus on only two or three criteria.

On the page before a student work sample, we provide comments on some of the solution criteria that are evident from the student's solution. The professional development experience described below will help the teacher use the solution guide in the classroom and will also help guide the teacher to use other assessment evaluation tools.

### **Where can I get a copy of *Geometry Assessments*, second edition?**

Look for the *Geometry Assessments* in book or CD-ROM format starting in Summer 2007 in our online catalog at [www.utdanacenter.org/catalog](http://www.utdanacenter.org/catalog). At that time, the second edition of *Geometry Assessments*, or a portion thereof, will also be available for download from the Dana Center's mathematics toolkit at [www.mathtekstoolkit.org](http://www.mathtekstoolkit.org).

### **Is professional development available to support the *Geometry Assessments*?**

Yes. The Dana Center has developed a three-day TEXTEAMS institute that focuses on the implementation of the assessments: ***TEXTEAMS Practice-Based Professional Development: Geometry Assessments***.

#### **Participants in these institutes will**

- experience selected assessments,
- analyze student work to evaluate student understanding,
- consider methods for evaluating student work,
- view a video of students working on the assessments,
- develop strategies for classroom implementation, and
- consider how the assessments support the TAKS.

Contact your local school district or regional service center to find out when these institutes are offered.

## Geometry Solution Guide

Name of Student \_\_\_\_\_ Name of Problem \_\_\_\_\_

The teacher will mark the criteria to be considered in the solution of this particular problem.

Mark the criteria to be considered in the solution of this particular problem.	<b>Criteria</b>	Check here if the solution satisfies this criteria.
	Identifies the important elements of the problem.	
	Shows an understanding of the relationships among elements.	
	Uses geometric and other mathematical principles to justify the reasoning used to analyze the problem.	
	Evaluates reasonableness or significance of the solution in the context of the problem.	
	Demonstrates geometric concepts, processes, and skills.	
	Makes an appropriate and accurate representation of the problem using correctly labeled diagrams.	
	Communicates clear, detailed, and organized solution strategy.	
	States a clear and accurate solution using correct units.	
	Uses appropriate terminology and notation.	
	Uses appropriate tools.	

## Assessment Alignment to TEKS and TAKS

Assessment Title	Page #	TEKS Focus	Additional TEKS	Connection to TAKS <sup>1</sup>
Cross-Country Cable		G.7.A, G.9.B	G.7.B, G.7.C	3, 7
Whitebeard's Treasure		G.3.B, G.7.B	G.3.D, G.3.E, G.7.A, G.7.C	3, 7
Quadrilateral Quandary		G.7.B, G.11.B	G.7.A, G.7.C, G.11.A	3, 4, 7, 8
Quite a Quilt		G.7.A, G.7.B	G.3.B, G.7.C, G.8.A	2, 3, 4, 7, 8
Wearable Art		G.2.B, G.7.B, G.7.C	G.2.A, G.7.A	3, 7
Sea Quest		G.7.A, G.5.A	NONE	4, 7
Pizza Delivery Service Regions		G.2.A	G.2.B, G.3.D	6, 7
More Pizza Delivery		G.2.B, G.7.B	G.7.A, G.7.C, G.9.A, G.10.B	3, 6, 7
Conjecture as Discovery and Proof as Explanation		G.2.B, G.3.B, G.7.A	G.3.C, G.3.D, G.3.E, G.7.C	3, 6, 7
Extending the Triangle Midsegment Conjecture		G.3.B, G.3.E	G.3.C, G.3.D	6, 7, 8
Why Doesn't My Conjecture Always Work?		G.3.C, G.5.B	G.3.D, G.3.E, G.5.A, G.9.B	6
Steiner's Point		G.2.A, G.2.B	G.4, G.5.D, G.7.A	6, 7
Steiner's Point Revisited		G.2.A, G.5.C	G.2.B, G.4, G.5.D, G.7.A	6, 7
Walking the Archimedean Walk		G.1.B, G.5.B	G.5.A, G.5.D	6
Talking the Archimedean Talk		G.5.B, G.5.D	G.1.B, G.2.A, G.4, G.5.A, G.8.C	6
Mad as a Hatter or Hat as a Madder		G.3.A, G.3.C	G.1.A	
Going the Distance in Taxicab Land		G.1.A, G.1.C	G.7.A	7
Angle Bisectors and Parallel Lines		G.9.A	G.2.A	6, 7
Circles and Tangents		G.9.C	G.11.A	7
The Clubhouse		G.9.A	G.10.A	7
Diagonals and Polygons		G.5.B, G.9.B	G.2.B, G.3.D, G.5.A	1, 2, 6

<sup>1</sup> The majority of these assessments meet TAKS Objective 10. Objective 10 should not be taught or learned in isolation; it should be integrated throughout all areas of study in a Geometry course.

Assessment Title	Page #	TEKS Focus	Additional TEKS	Connection to TAKS <sup>1</sup>
The Most Juice		G.8.D, G.9.D	G.5.D, G.8.C	7, 8
Boxing Basketballs		G.5.D, G.8.D	G.8.C	6, 8
Flower		G.8.A, G.8.B	G.5.D, G.8.C	8
Great Pyramid		G.6.B, G.8.D	G.8.C, G.9.D	7, 8
Walter and Juanita's Water Troughs		G.5.D, G.8.D	G.8.A, G.8.C	8
Greenhouse		G.8.A, G.8.D	G.8.B	8
Nesting Hexagons		G.5.A, G.8.A	G.5.B, G.5.D	1, 6, 8
Perfume Packaging		G.6.B, G.8.A, G.8.D	G.5.D, G.8.B	7, 8
Playing with Pipes		G.6.C, G.8.A	G.6.A	7, 8
Circular Security		G.6.B	G.6.C, G.8.C	7, 8
Different Views		G.6.C	G.6.B	7
The Slice is Right!		G.6.A	G.8.D	7, 8
Median to the Hypotenuse of a Right Triangle		G.2.B, G.3.B	G.1.A, G.2.A, G.3.A, G.3.C, G.10.A	6
Tell Me Everything You Can About. . .		G.9.B, G.10.B	G.1.A, G.2.A, G.3.B	6
Tiling with Four Congruent Triangles		G.3.B, G.10.A	G.2.B, G.3.C, G.5.C, G.9.B, G.10.B, G.11.A	6, 8
Shadow's Doghouse		G.8.A, G.8.D, G.11.D	G.1.A, G.6.C, G.10.A, G.10.B	6, 8
The School Flag		G.8.A, G.10.B	G.4, G.5.C, G.10.A	6, 8
The Shortest Cable Line		G.5.C, G.10.A	G.4	6
Ancient Ruins		G.11.A, G.11.B	G.5.B, G.11.C, G.11.D	8
Sightseeing Walk		G.11.A, G.11.B	G.5.B, G.8.C, G.11.C	6, 8
Will It Fit?		G.11.B, G.11.C	G.5.D, G.8.C	8
Spotlights		G.11.A, G.11.B	G.7.A, G.7.B, G.11.C	7, 8

<sup>1</sup> The majority of these assessments meet TAKS Objective 10. Objective 10 should not be taught or learned in isolation; it should be integrated throughout all areas of study in a Geometry course.



## TEKS and TAKS Alignment to Assessment

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G.3.B, G.7.B	G.3.D, G.3.E, G.7.A, G.7.C	3, 7	Whitebeard's Treasure	
G.7.B, G.11.B	G.7.A, G.7.C, G.11.A	3, 4, 7, 8	Quadrilateral Quandary	
G.7.A, G.7.B	G.3.B, G.7.C, G.8.A	2, 3, 4, 7, 8	Quite a Quilt	
G.2.B, G.7.B, G.7.C	G.2.A, G.7.A	3, 7	Wearable Art	
G.7.A, G.5.A	NONE	4, 7	Sea Quest	
G.2.A	G.2.B, G.3.D	6, 7	Pizza Delivery Service Regions	
G.2.B, G.7.B	G.7.A, G.7.C, G.9.A, G.10.B	3, 6, 7	More Pizza Delivery	
G.2.B, G.3.B, G.7.A	G.3.C, G.3.D, G.3.E, G.7.C	3, 6, 7	Conjecture as Discovery and Proof as Explanation	
G.3.B, G.3.E	G.3.C, G.3.D	6, 7, 8	Extending the Triangle Midsegment Conjecture	
G.3.C, G.5.B	G.3.D, G.3.E, G.5.A, G.9.B	6	Why Doesn't My Conjecture Always Work?	
G.2.A, G.2.B	G.4, G.5.D, G.7.A	6, 7	Steiner's Point	
G.2.A, G.5.C	G.2.B, G.4, G.5.D, G.7.A	6, 7	Steiner's Point Revisited	
G.1.B, G.5.B	G.5.A, G.5.D	6	Walking the Archimedean Walk	
G.5.B, G.5.D	G.1.B, G.2.A, G.4, G.5.A, G.8.C	6	Talking the Archimedean Talk	
G.3.A, G.3.C	G.1.A		Mad as a Hatter or Hat as a Madder	
G.1.A, G.1.C	G.7.A	7	Going the Distance in Taxicab Land	
G.9.A	G.2.A	6, 7	Angle Bisectors and Parallel Lines	
G.9.C	G.11.A	7	Circles and Tangents	
G.9.A	G.10.A	7	The Clubhouse	
G.5.B, G.9.B	G.2.B, G.3.D, G.5.A	1, 2, 6	Diagonals and Polygons	

<sup>1</sup> The majority of these assessments meet TAKS Objective 10. Objective 10 should not be taught or learned in isolation; it should be integrated throughout all areas of study in a Geometry course.

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G.5.D, G.8.D	G.8.C	6, 8	Boxing Basketballs	
G.8.A, G.8.B	G.5.D, G.8.C	8	Flower	
G.5.D, G.8.D	G.8.A, G.8.C	8	Walter and Juanita's Water Troughs	
G.6.B, G.8.D	G.8.C, G.9.D	7, 8	Great Pyramid	
G.8.A, G.8.D	G.8.B	8	Greenhouse	
G.5.A, G.8.A	G.5.B, G.5.D	1, 6, 8	Nesting Hexagons	
G.6.B, G.8.A, G.8.D	G.5.D, G.8.B	7, 8	Perfume Packaging	
G.6.C, G.8.A	G.6.A	7, 8	Playing with Pipes	
G.6.B	G.6.C, G.8.C	7, 8	Circular Security	
G.6.C	G.6.B	7	Different Views	
G.6.A	G.8.D	7, 8	The Slice is Right!	
G.2.B, G.3.B	G.1.A, G.2.A, G.3.A, G.3.C, G.10.A	6	Median to the Hypotenuse of a Right Triangle	
G.9.B, G.10.B	G.1.A, G.2.A, G.3.B	6	Tell Me Everything You Can About ...	
G.3.B, G.10.A	G.2.B, G.3.C, G.5.C, G.9.B, G.10.B, G.11.A	6, 8	Tiling with Four Congruent Triangles	
G.8.A, G.8.D, G.11.D	G.1.A, G.6.C, G.10.A, G.10.B	6, 8	Shadow's Doghouse	
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G.5.C, G.10.A	G.4	6	The Shortest Cable Line	
G.11.A, G.11.B	G.5.B, G.11.C, G.11.D	8	Ancient Ruins	
G.11.A, G.11.B	G.5.B, G.8.C, G.11.C	6, 8	Sightseeing Walk	
G.11.B, G.11.C	G.5.D, G.8.C	8	Will It Fit?	
G.11.B, G.11.C	G.7.A, G.7.B, G.11.C	7, 8	Spotlights	

<sup>1</sup> The majority of these assessments meet TAKS Objective 10. Objective 10 should not be taught or learned in isolation; it should be integrated throughout all areas of study in a Geometry course.

**Chapter 1:**  
*Coordinate  
Geometry*



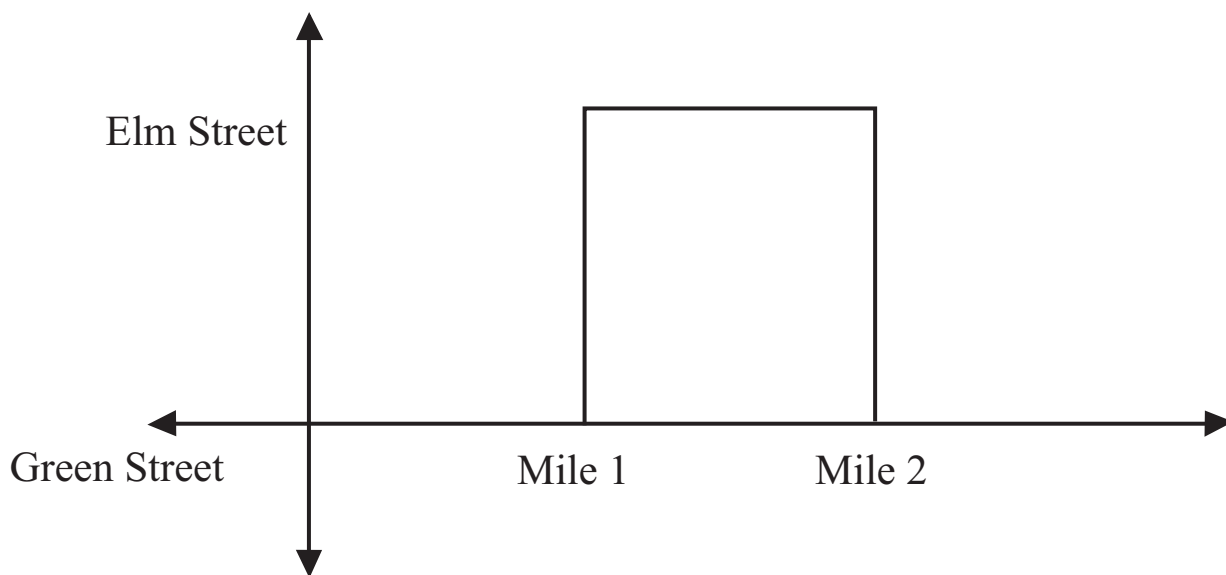
# Introduction

The assessments in this chapter use the coordinate system as a convenient and efficient way of representing geometric figures and investigating geometric relationships. These performance tasks provide the connections between what students learned in Algebra I and the geometric concepts. For example, Whitebeard's Treasure may be used as a performance task to assess the student's knowledge of algebraic concepts and concepts learned about geometric figures in earlier mathematics courses.



## Cross-Country Cable

Jose owns a square parcel of land on the north side of Green Street. The property is on the edge of the road between the one- and two-mile markers. Elm Street runs perpendicular to Green Street and one mile west of the one-mile marker.



The cable company wants to run a cable through Jose's property from the intersection of Elm Street and Green Street. The cable must divide Jose's parcel of land into two parts that are equal in area.

1. Determine the equation of the line that will represent the path of the buried cable.
2. Determine the length of the portion of the cable that runs across Jose's property.

## Notes

### Materials:

One graphing calculator per student

### Geometry TEKS Focus:

(G.7) **Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student is expected to:

(A) use one- and two-dimensional coordinate systems to represent points, lines, rays, line segments, and figures;

(G.9) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student is expected to:

(B) formulate and test conjectures about the properties and attributes of polygons and their component parts based on explorations and concrete models;

### Additional Geometry TEKS:

(G.7) **Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student is expected to:

(B) use slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons; and

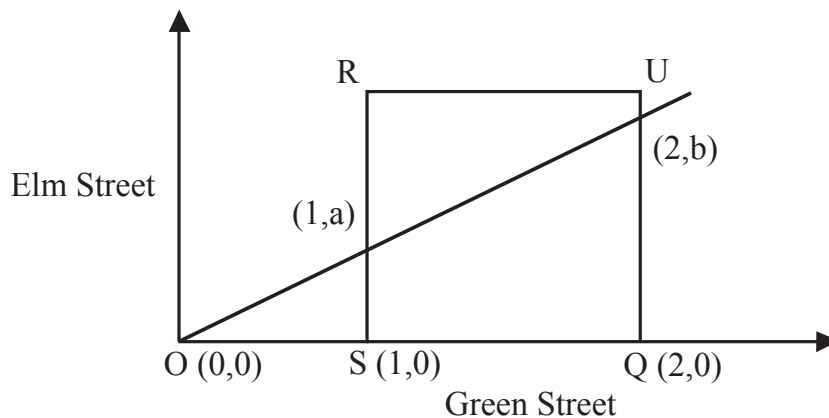
## Teacher Notes

### Scaffolding Questions:

- What information do you know about the roads near Jose's property and the location of his parcel of land?
- What needs to be added to the given figure before the equation of the line can be found?
- Where will the cable line originate? How can you label this point?
- In how many different ways might the cable intersect his property?
- Which way must the cable be drawn to divide it into two portions that are equal in area?
- How can you label the points where the cable will cross the boundaries of Jose's property?
- How can you determine the slope of the cable line?

### Sample Solutions:

1. Draw a figure in the coordinate plane to represent Jose's land.



Since  $SQRU$  is a square, the coordinates at  $R$  are  $(1,1)$ , and the coordinates at  $U$  are  $(2,1)$ . The coordinates of the



intersection of  $\overline{SR}$  and the cable line are  $(1,a)$ , and the point of intersection of  $\overline{UQ}$  and the cable line are  $(2,b)$ .

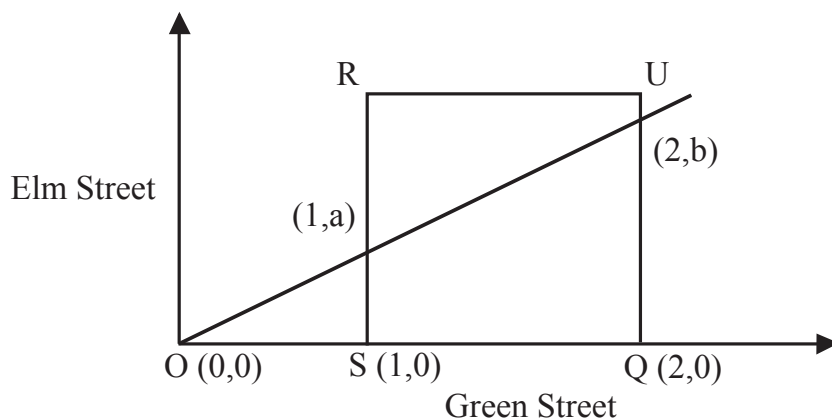
The cable originated at the intersection of Elm Street and Green Street. This point is the origin. The cable is going to begin at the origin. Therefore, one point on the line is  $(0,0)$ . The  $y$ -intercept of the line will also be 0. The line will also contain point  $(1,a)$ .

The slope of the line is  $\frac{a-0}{1-0} = a$ . The equation of this line is  $y = ax$ .

Using points  $(2,b)$  and  $(1,a)$ , the slope of the line can also be expressed as  $\frac{b-a}{2-1} = b-a$ .

When the two slopes are set equal to each other,  $a = b - a$ , the equation can be simplified into  $2a = b$  or  $b = 2a$ .

The cable divides the area of the square into two equal parts. Each upper and lower area is in the shape of a trapezoid.



We know that the area of the land is 1 square mile, and the area of one trapezoid is  $\frac{1}{2}$  square mile. Using this information the area of one trapezoid can be calculated, and the numerical value of the slope can be found.

The lower trapezoid (below the cable line) has vertical bases with lengths  $a$  and  $b$ . The height of the trapezoid is the length of the side running along Green Street. Therefore, the height of the trapezoid is 1 since we know the side length of the square is 1 mile. The area of the trapezoid is

(C) derive and use formulas involving length, slope, and midpoint.

**Connections to TAKS:**

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

**One teacher says . . .**

“To introduce the problem to the students, go over the criteria on the solution guide and allow students to read and decide how to solve the problem. Tell the students to read the problem silently before working as a group—it made it easier to work.”

found by multiplying one-half by the product of the sum of the bases and the height.

$$A = \frac{1}{2}(a+b) \cdot 1 = \frac{a+b}{2}$$

We know this area is equal to one-half of the area of the mile square, therefore

$$\begin{aligned} \frac{1}{2} &= \frac{a+b}{2} \\ a+b &= 1 \end{aligned}$$

By substituting the slope  $b = 2a$  into the area  $1 = a + b$ , we find that

$$\begin{aligned} 1 &= a + 2a \\ 1 &= 3a \\ \frac{1}{3} &= a \end{aligned}$$

The value of the slope for the equation  $y = ax$  is  $\frac{1}{3}$ , so the equation of the cable line that bisects Jose's property is  $y = \frac{1}{3}x$ .

2. The length of the cable is represented by the distance from  $(1,a)$  to  $(2,b)$ . The value of  $a = \frac{1}{3}$ , and the value of  $b = 2 \cdot \frac{1}{3} = \frac{2}{3}$ .

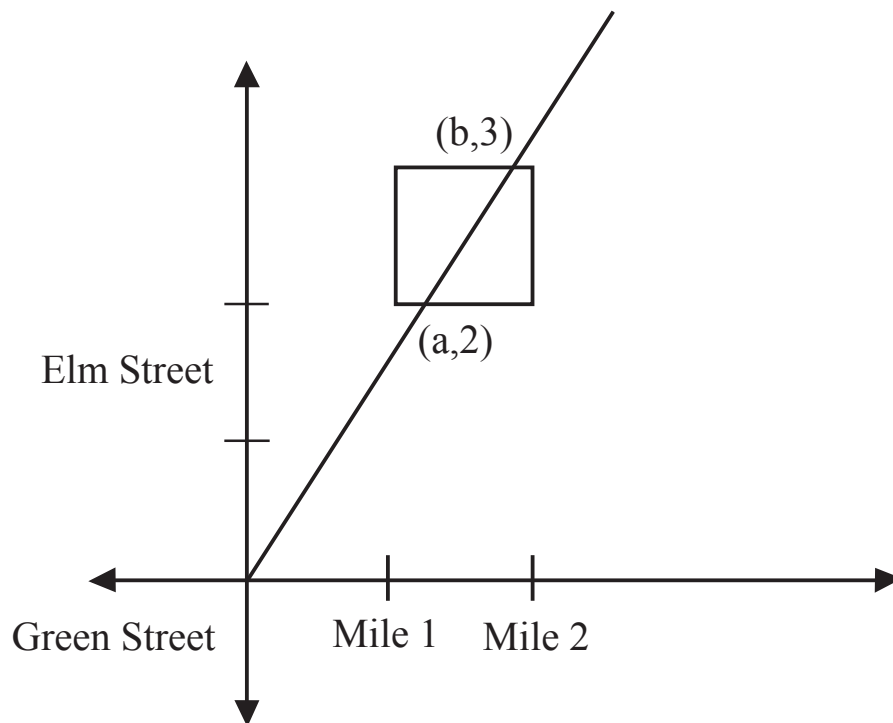
The coordinates of the vertices representing the opposite corners of Jose's land are  $(1, \frac{1}{3})$  and  $(2, \frac{2}{3})$ . In other words, these are the points where the cable intersects the land parcel.

The length of the portion of cable running across the property can be found using the distance formula:

$$\begin{aligned} \sqrt{(2-1)^2 + \left(\frac{2}{3} - \frac{1}{3}\right)^2} &= \sqrt{(1)^2 + \left(\frac{1}{3}\right)^2} = \sqrt{\frac{10}{9}} = \frac{\sqrt{10}}{3} \text{ miles} \\ &\approx 1.05 \text{ miles} \end{aligned}$$

**Extension Question:**

- Describe how this situation would change if the one-square-mile parcel of land were two miles north of Green Street and one mile east of Elm Street. Justify your description with a diagram, and show your work.



The slope of the line can be represented as

$$\frac{3 - 0}{b - 0} = \frac{3}{b} \text{ or}$$

$$\frac{2 - 0}{a - 0} = \frac{2}{a}$$

$$\frac{3}{b} = \frac{2}{a}$$

$$b = \frac{3}{2}a$$

Consider the portion to the left of the line. The height of the square is 1 unit. The two bases of the trapezoid may be expressed as  $a - 1$  and  $b - 1$ . The area of the left half of the square parcel of land can be represented in this way:

$$\frac{1}{2} = \frac{1}{2}[(a-1) + (b-1)] \cdot 1$$

$$1 = a + b - 2$$

$$3 = a + b$$

Substituting  $\frac{3}{2}a$  for  $b$ , we get

$$3 = a + \frac{3}{2}a$$

$$3 = \frac{5}{2}a$$

$$a = \frac{6}{5}$$

Now, substituting for  $a$ ,

$$b = \frac{3}{2}a = \frac{3}{2}\left(\frac{6}{5}\right) = \frac{9}{5}$$

The points are  $\left(\frac{6}{5}, 2\right)$  and  $\left(\frac{9}{5}, 3\right)$ .

The slope of the line is  $\frac{3 - 2}{\left(\frac{9}{5} - \frac{6}{5}\right)} = \frac{1}{\frac{3}{5}} = \frac{5}{3}$ .

The equation of the line is  $y = \frac{5}{3}x$ .

To verify that the line divides the rectangle into two polygons of equal area, determine the area of the two trapezoids.

Left trapezoid:

$$\frac{1}{2}\left(\frac{6}{5} - 1 + \frac{9}{5} - 1\right) \cdot 1 = \frac{1}{2}\left(\frac{15}{5} - 2\right) = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

Right trapezoid:

$$\frac{1}{2}\left(2 - \frac{6}{5} + 2 - \frac{9}{5}\right) \cdot 1 = \frac{1}{2}\left(4 - \frac{15}{5}\right) = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

The land has been divided into two equal portions by the line  $y = \frac{5}{3}x$ .



## Whitebeard's Treasure

Whitebeard, the notorious pirate of the West Bay, buried treasure on Tiki Island over 200 years ago. Archeologists recently discovered a map showing the location of the treasure. The location has generated quite a bit of media attention, much to the dismay of the archeologists. In order to allow both the media and archeologists to work together, officials have decided to erect two fences around the location, allowing the media access to the site, yet allowing the archeologists room to work. One fence encloses the actual area where the archeologists will work. Another fence surrounds the enclosed dig area.

Descriptions of the fencing locations have been provided to the media so they may indicate accessible areas for their employees. Use the given information to draw and label a quadrilateral on graph paper indicating the location of the two fences.

1. Corners of the first fence are located at points  $A(11,3)$ ,  $B(3,-11)$ ,  $C(-13,-9)$  and  $D(-5,9)$ . The media must stay within this fenced area. Connect the points in alphabetical order, and then join point D to Point A.
2. Find and label the midpoints of each segment of quadrilateral ABCD, showing all work. Label the midpoints of the segments as follows:

$\overline{AB}$  has midpoint Q,

$\overline{BC}$  has midpoint R,

$\overline{CD}$  has midpoint S,

$\overline{DA}$  has midpoint T.

3. Connect the four midpoints in alphabetical order to create a new quadrilateral QRST. This quadrilateral represents the fence surrounding the archeological dig site.
4. Quadrilateral ABCD was an ordinary quadrilateral, but QRST is a special one. Determine the special name for quadrilateral QRST, and justify your answer using coordinate geometry in two different ways.

## Notes

### Materials:

One graphing calculator per student

### Geometry TEKS Focus:

(G.3) **Geometric structure.** The student applies logical reasoning to justify and prove mathematical statements.

The student is expected to:

(B) construct and justify statements about geometric figures and their properties;

### (G.7) Dimensionality and the geometry of location.

The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

(B) use slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons; and

### Additional Geometry TEKS:

(G.3) **Geometric structure.** The student applies logical reasoning to justify and prove mathematical statements.

The student is expected to:

(D) use inductive reasoning to formulate a conjecture; and

(E) use deductive reasoning to prove a statement.

### (G.7) Dimensionality and the geometry of location.

The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

## Teacher Notes

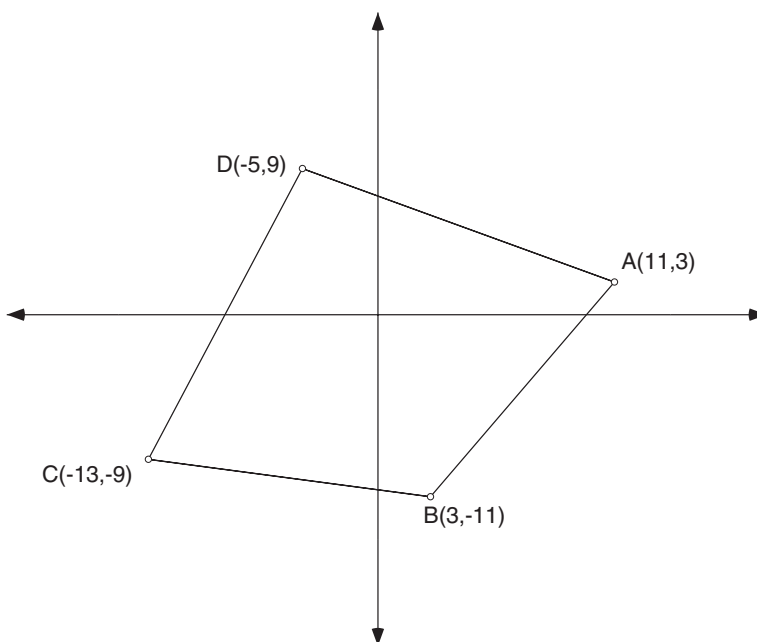
This performance task addresses the same mathematical concepts as the Wearable Art task later in this chapter. Whitebeard's Treasure gives the numerical coordinates. In Wearable Art the coordinates are given and the student must represent the situation using variable coordinates for the points. The teacher may choose to use one or both of these problems.

### Scaffolding Questions:

- What is the formula for finding the midpoint of a line segment?
- Which of the quadrilaterals are special quadrilaterals?
- What are the characteristics of each special quadrilateral?
- What characteristics does quadrilateral QRST appear to possess that matches one of the special quadrilaterals?
- How can you prove these special characteristics?

### Sample Solutions:

Quadrilateral ABCD is graphed as shown. This is the outer fence.





To find the midpoint of each segment of quadrilateral ABCD, use the midpoint formula.

The midpoint of the segment with endpoints  $(x_1, y_1)$  and  $(x_2, y_2)$  is  $\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$ .

To find the midpoint of each segment, substitute the  $x$  and  $y$  values from the endpoints of the segment into the formula as follows:

Midpoint of  $\overline{AB}$  (Point Q)

$$\left(\frac{11 + 3}{2}, \frac{3 + (-11)}{2}\right) = (7, -4)$$

Midpoint of  $\overline{BC}$  (Point R)

$$\left(\frac{3 + (-13)}{2}, \frac{-11 + (-9)}{2}\right) = (-5, -10)$$

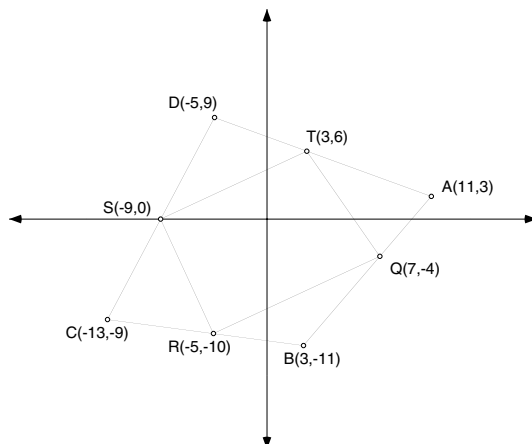
Midpoint of  $\overline{CD}$  (Point S)

$$\left(\frac{-13 + (-5)}{2}, \frac{-9 + 9}{2}\right) = (-9, 0)$$

Midpoint of  $\overline{DA}$  (Point T)

$$\left(\frac{-5 + 11}{2}, \frac{9 + 3}{2}\right) = (3, 6)$$

Graph the midpoints and connect them in alphabetical order to form a new quadrilateral QRST.



The student is expected to:

(A) use one- and two-dimensional coordinate systems to represent points, lines, rays, line segments, and figures;

(C) derive and use formulas involving length, slope, and midpoint.

**Connections to TAKS:**

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

**One teacher says . . .**

“I had the students get out their notes before they started. I gave them two minutes to read the problem and think of questions they wanted to ask the class or me. Some of the scaffolding questions I asked were:

What types of quadrilaterals are there?

How do you know what type of quadrilateral it is?

How do you know when two lines are parallel?

How do you know when two lines are perpendicular?”

Quadrilateral QRST would be the fence that encloses the archeologists' dig site.

Quadrilateral QRST appears to be a parallelogram because the opposite sides of the newly formed quadrilateral appear to be parallel. One way to prove that a quadrilateral is a parallelogram is to prove that both pairs of opposite sides are parallel. Lines that have the same slope are parallel lines.

Use the slope formula:

The slope of the line through points  $(x_1, y_1)$  and  $(x_2, y_2)$  is  $m = \frac{y_2 - y_1}{x_2 - x_1}$ .

The slope of  $\overleftrightarrow{RS}$  is  $\frac{0 - (-10)}{-9 - (-5)} = \frac{10}{-4} = -\frac{5}{2}$

The slope of  $\overleftrightarrow{QT}$  is  $\frac{6 - (-4)}{3 - 7} = \frac{10}{-4} = -\frac{5}{2}$

$RS \parallel QT$  because both lines have the same slope.

The slope of  $\overleftrightarrow{ST}$  is  $\frac{6 - 0}{3 - (-9)} = \frac{6}{12} = \frac{1}{2}$

The slope of  $\overleftrightarrow{RQ}$  is  $\frac{-4 - (-10)}{7 - (-5)} = \frac{6}{12} = \frac{1}{2}$

$ST \parallel RQ$  because both lines have the same slope.

Quadrilateral QRST is a parallelogram by definition because both pairs of opposite sides are parallel.

Another way to show that QRST is a parallelogram is to prove that both sides of opposite sides are congruent (using the distance formula to find the lengths of each side).

$$QR = \sqrt{(7 - (-5))^2 + (-4 - (-10))^2} = \sqrt{144 + 36} = \sqrt{180} = 6\sqrt{5}$$

$$ST = \sqrt{(-9 - 3)^2 + (0 - 6)^2} = \sqrt{144 + 36} = \sqrt{180} = 6\sqrt{5}$$

$$QR = ST = 6\sqrt{5}$$

$$RS = \sqrt{(-5 - (-9))^2 + (-10 - 0)^2} = \sqrt{16 + 100} = \sqrt{116} = 2\sqrt{29}$$

$$QT = \sqrt{(7 - 3)^2 + (-4 - 6)^2} = \sqrt{16 + 100} = \sqrt{116} = 2\sqrt{29}$$

$$RS = QT = 2\sqrt{29}$$

Both pairs of opposite sides of quadrilateral QRST are congruent. Therefore, it is a parallelogram.

**Extension Questions:**

- Use algebra to find the point of intersection of the diagonals of quadrilateral QRST.

To find the point where the diagonals intersect, the equations of lines  $\overleftrightarrow{RT}$  and  $\overleftrightarrow{SQ}$  must be identified and then used to find the point of intersection.

The slope of  $\overleftrightarrow{RT}$  is  $\frac{6 - (-10)}{3 - (-5)} = \frac{16}{8} = 2$

The slope of  $\overleftrightarrow{SQ}$  is  $\frac{0 - (-4)}{-9 - 7} = \frac{4}{-16} = -\frac{1}{4}$

The equation of  $\overleftrightarrow{RT}$  is  $(y - 6) = 2(x - 3)$  or  $y = 2x$

The equation of  $\overleftrightarrow{SQ}$  is

$$y = -\frac{1}{4}(x - (-9))$$

$$y = -\frac{1}{4}x - \frac{9}{4}$$

The point where the diagonals intersect can be found by using linear combination.

$$y = 2x$$

$$y = -\frac{1}{4}x - \frac{9}{4}$$

$$2x = -\frac{1}{4}x - \frac{9}{4}$$

$$8x = -1x - 9$$

$$9x = -9$$

$$x = -1$$

$$y = 2x = 2(-1) = -2$$

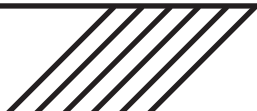
The point of intersection is  $(-1, -2)$ .

- Use coordinate geometry to prove the diagonals of quadrilateral QRST bisect each other.

The midpoint of  $\overline{QS}$  is  $\left(\frac{7+(-9)}{2}, \frac{4+0}{2}\right) = (-1, -2)$

The midpoint of  $\overline{RT}$  is  $\left(\frac{-5+3}{2}, \frac{-10+6}{2}\right) = (-1, -2)$

The midpoints of the segment are the same point as the intersection point. The diagonals bisect each other.



## Student Work Sample

Field Test Teacher's Comment:

This was the first performance task I had the students do using a poster. I enjoyed the poster and I feel most of the students did, too. I would like to have done this problem during the quadrilaterals section and will do so next year. One thing I did different this time was to have the students write on the back of their solution guide exactly what to put on their posters for the three criteria we emphasized.

The teacher emphasized three criteria from the Geometry Solution Guide found in the introduction to this book. On the back of one student's solution guide were these notes:

Shows an understanding of the relationships among elements

- Statement showing how the elements are related.
- Can the history teacher understand your steps?

Makes an appropriate and accurate representation of the problem using correctly labeled diagrams

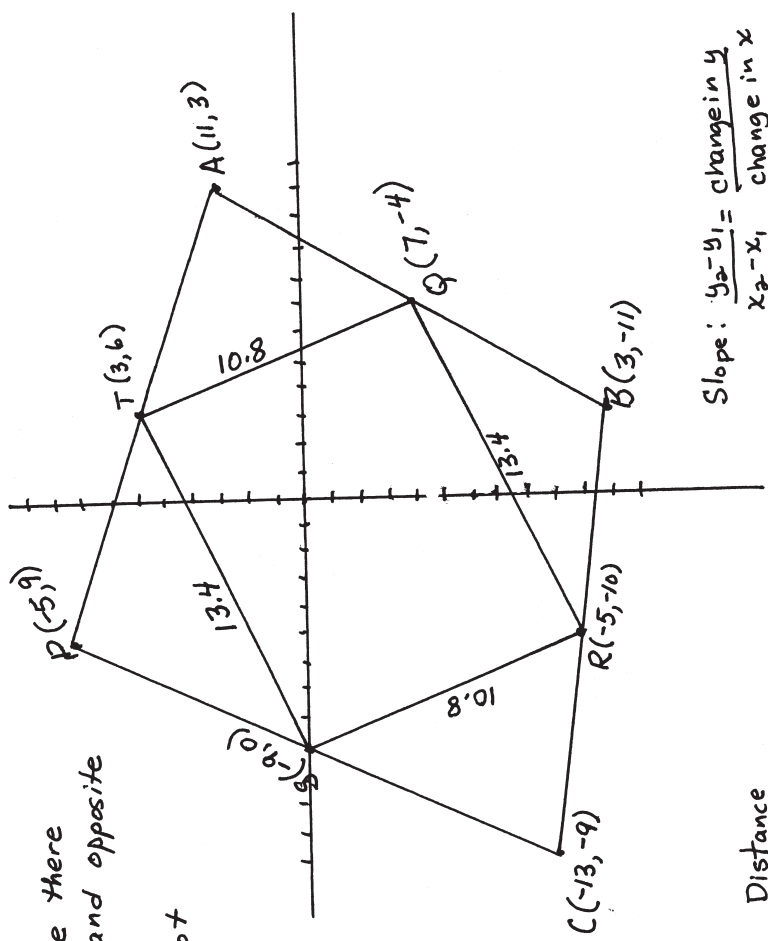
- Drawing the pictures
- Make appropriate markings on the picture

Communicates clear, detailed, and organized solution strategy

- Step by step details that can be followed
- Don't plug in a number without showing why/how
- Must have justification
- Explaining your thinking!!!

A copy of the poster from this student's group appears on the next page.

A parallelogram - because there are two parallel sides and opposite sides are equal.  
 The diagonals are also not equal so it is not a rectangle or a square.



Midpoints on graph.

$$RS = \sqrt{(-9-5)^2 + (0-70)^2} = \sqrt{16+100} = \sqrt{116} \approx 10.8$$

$$TQ = \sqrt{(7-3)^2 + (-4-6)^2} = \sqrt{16+100} = \sqrt{116} \approx 10.8$$

Opposite sides are equal.  
 $RS = TQ$

$$QR = \sqrt{(-5-7)^2 + (-10-4)^2} = \sqrt{144+36} = \sqrt{180} \approx 13.4$$

$$ST = \sqrt{(-9-3)^2 + (0-6)^2} = \sqrt{144+36} = \sqrt{180} \approx 13.4$$

Opposite sides are equal.  
 $QR = ST$

Distance

$$RT = \sqrt{(-5-3)^2 + (-10-6)^2} = \sqrt{64+256} = \sqrt{320} \approx 17.9$$

$$SQ = \sqrt{(-9-7)^2 + (0-4)^2} = \sqrt{256+16} = \sqrt{272} \approx 16.5$$

The diagonals are not equal.

Slope:  $\frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x}$

$$QR: \frac{10-4}{-5-7} = \frac{-6}{-12} = \frac{1}{2} \quad \text{because the slopes are the same.}$$

$$ST: \frac{6-0}{3-9} = \frac{6}{-9} = \frac{2}{-3} = \frac{1}{2} \quad \text{because the slopes are the same.}$$

$$RS: \frac{0-10}{-9-5} = \frac{-10}{-14} = \frac{5}{7}$$

$$QT: \frac{6-4}{3-7} = \frac{2}{-4} = \frac{1}{-2} = -\frac{1}{2}$$

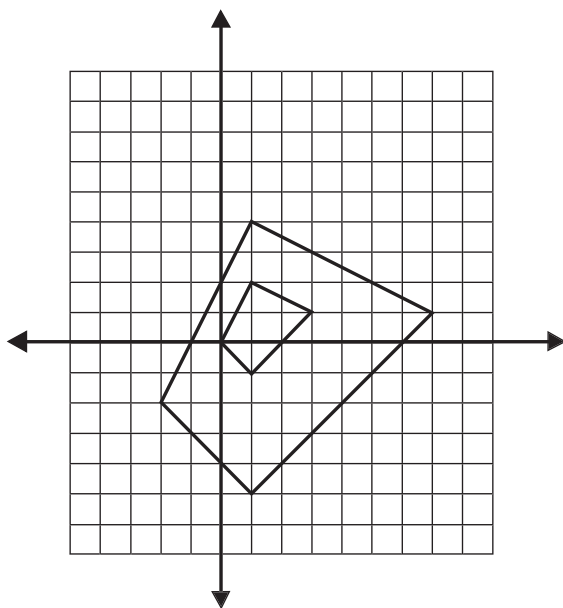




## Quadrilateral Quandary

The Seaside Hotel is going to install new landscaping on the hotel grounds. Plans that included the design and amounts of exact materials needed were purchased immediately after the landscaping project was approved. Due to recent water restrictions, however, the board of directors has decided to modify the project plans so that a smaller landscape bed will be constructed.

The hotel's landscape engineer has modified the size of the bed as shown on the graph below. The large trapezoid is the original plan, and the smaller trapezoid is the new plan. She must calculate the amount of the size reduction so that a detailed mathematical explanation can be presented to the board of directors and the new costs can be calculated.



1. You have been hired to help the landscape engineer prepare her report and have been instructed to justify all of your calculations. You remember that the center of dilation is the intersection of two or more lines, each containing a point from the original figure and the corresponding point from the dilated figure. Use this definition and the figures on the graph to calculate the center of dilation.
2. Find the scale factor for this dilation using the center of dilation and corresponding points of the quadrilaterals. Explain your process.
3. Verify that your scale factor is correct using the distance formula and corresponding parts of the figure.

## Notes

### Materials:

One graphing calculator per student

### Geometry TEKS Focus:

#### (G.7) Dimensionality and the geometry of location.

The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student is expected to:

(B) use slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons; and

#### (G.11) Similarity and the geometry of shape.

The student applies the concepts of similarity to justify properties of figures and solve problems.

The student is expected to:

(B) use ratios to solve problems involving similar figures;

### Additional Geometry TEKS:

#### (G.7) Dimensionality and the geometry of location.

The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student is expected to:

(A) use one- and two-dimensional coordinate systems to represent points, lines, rays, line segments, and figures;

(C) derive and use formulas involving length, slope, and midpoint.

## Teacher Notes

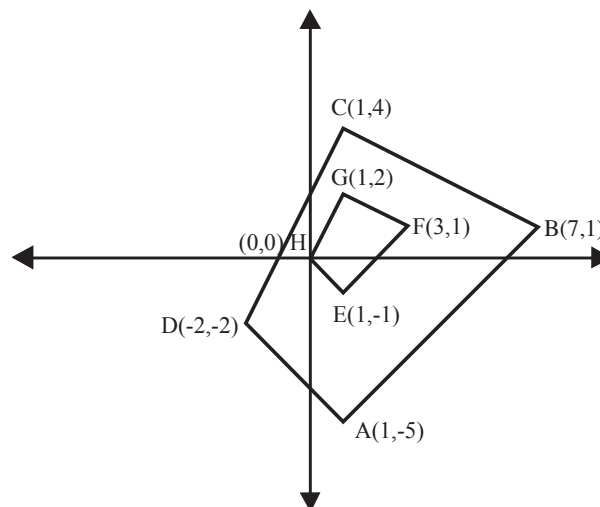
### Scaffolding Questions:

- What kind of transformation is illustrated?
- Why is it necessary to calculate the center of dilation?
- Is the transformation rigid? Why or why not?
- What does the scale factor tell you about the size of the new figure?
- How does the value of the scale factor relate to the center of dilation?
- How do the dimensions of the new figure relate to the scale factor?

### Sample Solutions:

1. The center of dilation is the intersection of 2 or more lines, each containing a point from the original figure and a corresponding point from the dilated figure. The strategy used will be to find the equations of the intersecting lines, enter them into the graphing calculator, and find the point of intersection. That point of intersection will be the center of dilation.

First, label the coordinates of the vertices of the quadrilaterals as follows: A(1,-5); B(7,1); C(1,4); D(-2,2) and E(1,-1); F(3,1); G(1,2); H(0,0).



Using the point-slope formula, find the equation of the line that contains points A(1,-5) and E(1,-1) and an equation of the line containing D(-2,-2) and H(0,0).

The calculation for the slope of line  $\overleftrightarrow{DH}$  is  $1: \frac{-2 - 0}{-2 - 0} = 1$ .

The equation of line  $\overleftrightarrow{DH}$  is found using the point-slope formula with H:

$$y - 0 = 1(x - 0)$$

$$y = x$$

The line  $\overleftrightarrow{AE}$  is the vertical line  $x = 1$ . This line also passes through points C and G.

To find the intersection point of the line  $y = x$  and  $x = 1$  substitute 1 for  $x$ .

$$y = 1$$

The intersection point of the two lines is (1,1). The vertical line through points B and F is  $y = 1$ . This line also passes through the point (1,1).

The center of dilation is (1,1). Label that point X on the diagram.

- The center of dilation and the points A(1,-5) and E(1,-1) can be used in determining the scale factor. The ratio of the distances between the center of dilation and the corresponding points on the quadrilateral determines the scale factor. To calculate the scale factor, solve the equation:

$$\text{scale factor} = \frac{XE}{XA}$$

The distance formula is used to find the length of the segments.

$$XE = \sqrt{(1-1)^2 + (1-(-1))^2} = \sqrt{0+2^2} = 2$$

(G.11) **Similarity and the geometry of shape.** The student applies the concepts of similarity to justify properties of figures and solve problems.

The student is expected to:

(A) use and extend similarity properties and transformations to explore and justify conjectures about geometric figures; and

**Connections to TAKS:**

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 4: The student will formulate and use linear equations and inequalities.

Objective 7: The student will demonstrate an understanding of two-and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

The length of  $\overline{XA}$  is calculated as follows:

$$XA = \sqrt{(1-1)^2 + (1-(-5))^2} = \sqrt{0+6^2} = 6$$

$$\text{scale factor} = \frac{XE}{XA} = \frac{2}{6} = \frac{1}{3}$$

3. Using corresponding sides of the figures and the distance formula, verify the scale factor.

$$DA = \sqrt{(-2-1)^2 + (-2-(-5))^2} = \sqrt{(-3)^2 + (-3)^2} = \sqrt{18}$$

$$HE = \sqrt{(0-1)^2 + (0-(-1))^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

The ratio of the lengths of sides is

$$\frac{HE}{DA} = \frac{\sqrt{2}}{\sqrt{18}} = \sqrt{\frac{2}{18}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$$

$$DC = \sqrt{(-2-1)^2 + (-2-4)^2} = \sqrt{(-3)^2 + (-6)^2} = \sqrt{45}$$

$$HG = \sqrt{(0-1)^2 + (0-2)^2} = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$$

The ratio of the lengths of sides is

$$\frac{HG}{DC} = \frac{\sqrt{5}}{\sqrt{45}} = \sqrt{\frac{5}{45}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$$

$$CB = \sqrt{(-1-7)^2 + (4-1)^2} = \sqrt{(-6)^2 + (3)^2} = \sqrt{45}$$

$$GF = \sqrt{(1-3)^2 + (2-1)^2} = \sqrt{(-2)^2 + (1)^2} = \sqrt{5}$$

The ratio of the lengths of sides is

$$\frac{GF}{CB} = \frac{\sqrt{5}}{\sqrt{45}} = \sqrt{\frac{5}{45}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$$

$$AB = \sqrt{(1-7)^2 + (-5-1)^2} = \sqrt{(-6)^2 + (-6)^2} = \sqrt{72}$$

$$EF = \sqrt{(1-3)^2 + (-1-1)^2} = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8}$$

The ratio of the lengths of sides is

$$\frac{EF}{AB} = \frac{\sqrt{8}}{\sqrt{72}} = \sqrt{\frac{8}{72}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$$

Therefore, this scale factor is true for each pair of corresponding sides. Corresponding sides are  $\overline{DA}$  and  $\overline{HE}$ ,  $\overline{DC}$  and  $\overline{HG}$ ,  $\overline{CB}$  and  $\overline{GF}$ , and  $\overline{AB}$  and  $\overline{EF}$ .

**Extension Question:**

- Landscape timbers are sold in 8-foot lengths. Suppose the original project was designed for 850 feet of landscaping timber, at a cost of \$1.39 per timber. Calculate the minimum amount of materials that will need to be ordered for the new bed and the cost of materials before tax. Explain how you arrived at your answers.

The scale factor for the new project is  $\frac{1}{3}$  of the original design plan. If the original

project was designed for 850 feet of landscaping timber, the new project will require  $\frac{1}{3}$

as much.

$$\frac{1}{3} (850 \text{ feet}) = 283.3 \text{ feet.}$$

If the timbers are 8 feet long, divide 283.3 by 8. The result is 35.4125 timbers. Timbers are sold in 8-foot lengths; therefore, a minimum of 36 will be needed.

The cost of the project per landscape bed is calculated by multiplying \$1.39 by 36 timbers.

$$1.39 (36) = 50.04$$

The cost for each landscape bed will be \$50.04.

## Student Work Sample

The task analysis on the next page was created by a group of students. The students were given the task, told to work silently on it for at least three minutes, and asked to create a group poster of their analysis of the task.

The work is a good example of these criteria from the Geometry Solution Guide:

- Shows a relationship among the elements.

*Students demonstrated their understanding of which points corresponded to one another and used the appropriate labeling ( $A$ ,  $A'$ ).*

- Uses appropriate terminology and notation.

*Students used correct language for transformations (image, dilation, scale factor). They wrote and used the formulas for distance between two points.*

The student, however, neglected to give reasons for the steps and to explain the process for determining the scale factor.

$$\frac{A'B'}{AB} =$$

$$\frac{\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}}{\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}}$$

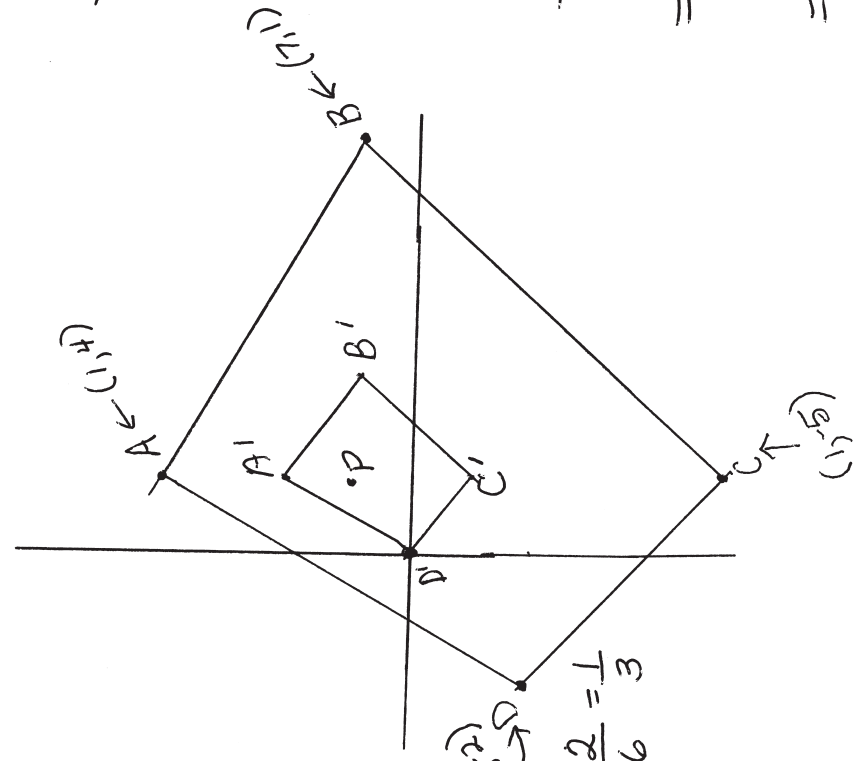
$$= \frac{\sqrt{(2-1)^2 + (1-3)^2}}{\sqrt{(4-1)^2 + (1-7)^2}}$$

$$= \frac{\sqrt{1^2 + (-2)^2}}{\sqrt{3^2 + (-6)^2}}$$

$$= \frac{\sqrt{5}}{\sqrt{45}} = \sqrt{\frac{5}{45}}$$

$$= \sqrt{\frac{1}{9}} = \frac{1}{3}$$

Quadrilateral Quandy



Original Image  
 A (1, 4) B (7, 1)  
 C (1, -5) D (-2, 2)

Revised Image  
 A' (1, 2) B' (3, 1)  
 C' (1, -1) D' (0, 0)

Center of Dilation  
 P (1, 1)

$$\frac{PB'}{PB} = SF \quad \frac{PB'}{PB} = \frac{2}{6} = \frac{1}{3}$$

Scale Factor =  $\frac{1}{3}$

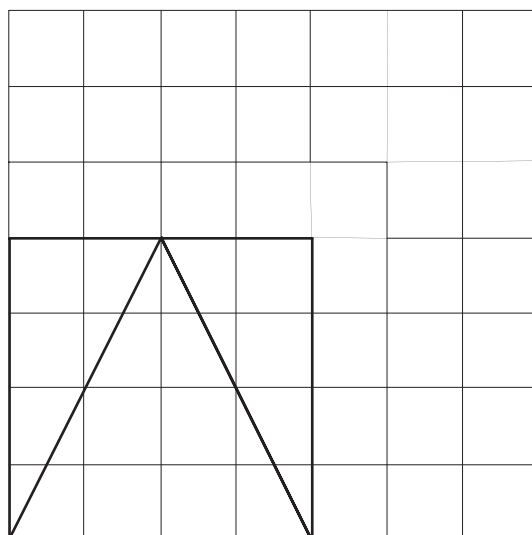




## Quite a Quilt

Ed has decided to enter a national geometry competition. The contest rules state that individuals must submit plans for a 4-inch quilt square design that will produce an octagonal region within the square whose area is no larger than 3 square inches. The entry must clearly explain and illustrate how the design is to be created, and it must prove that the inner octagonal area is within the contest guidelines.

Ed wants to design his square so that the midpoint of each side of the square is joined to its two opposite vertices. The figure below shows Ed's beginning sketch. Each square on the grid is equal to one square inch.



1. Complete the quilt design, and label the figure representing Ed's octagon.
2. Determine the coordinates of the vertices of the octagon. Justify your answer.
3. Explain how you know whether or not the octagon is a regular octagon.
4. Does his design meet the contest's area criteria as outlined above? Justify your solution using coordinate geometry, and show all your work.

**Notes**

**Materials:**

One graphing calculator per student

Dynamic geometry computer program (optional)

**Geometry TEKS Focus:**

**(G.7) Dimensionality and the geometry of location.**

The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student is expected to:

(A) use one- and two-dimensional coordinate systems to represent points, lines, rays, line segments, and figures;

(B) use slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons; and

**Additional Geometry TEKS:**

**(G.3) Geometric structure.**

The student applies logical reasoning to justify and prove mathematical statements.

The student is expected to:

(B) construct and justify statements about geometric figures and their properties;

**(G.7) Dimensionality and the geometry of location.**

The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

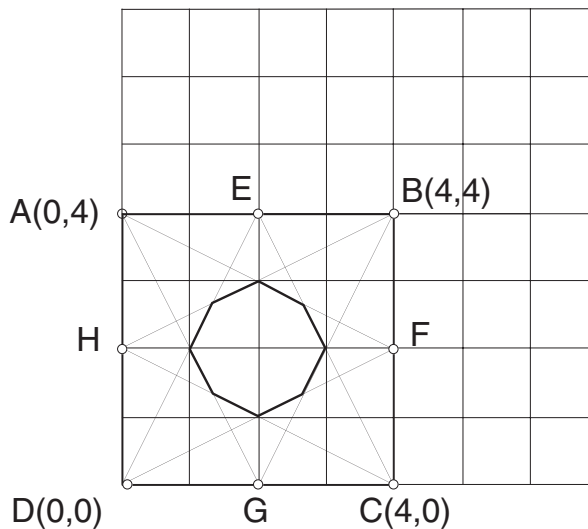
**Teacher Notes**

**Scaffolding Questions:**

- Describe how to determine the coordinates of each vertex of the square.
- Determine the coordinates of the midpoint of each side of the square.
- Determine the coordinates of the center of the square.
- How can you mathematically determine these points? (Note: Students may investigate this problem using a dynamic geometry computer program.)
- If you connect the vertices of the octagon to the center of the square, what do you know about the 8 triangles that are formed?
- What two properties must be satisfied if the octagon is to be a regular octagon?
- How could you determine the area of one-fourth of the octagon?

**Sample Solutions:**

1. The pattern created by joining the midpoint of each side of the square to its two opposite vertices is shown below.



The length of the side of the square is 4 inches. The coordinates of the midpoints will be:

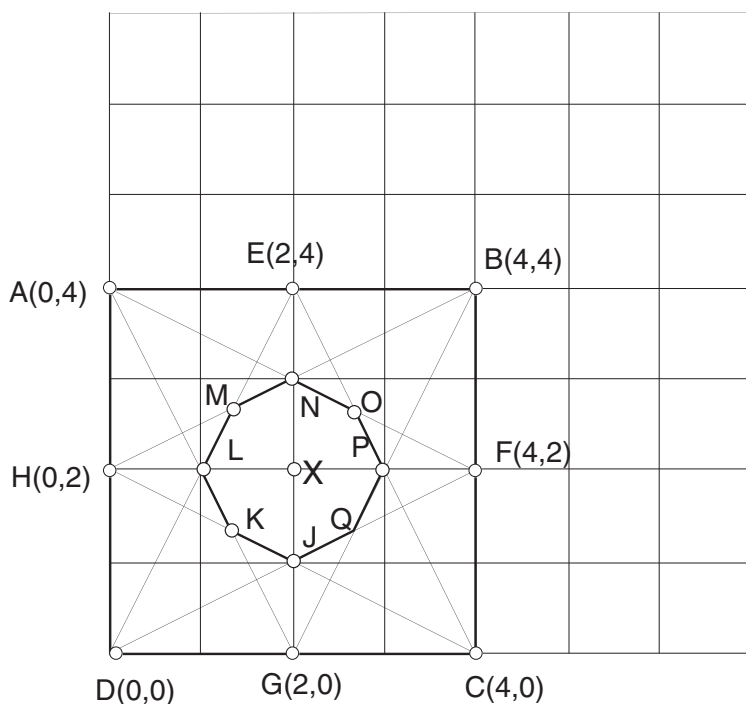
$$G = \left( \frac{0+4}{2}, \frac{0+0}{2} \right) = (2,0)$$

$$F = \left( \frac{4+4}{2}, \frac{0+4}{2} \right) = (4,2)$$

$$E = \left( \frac{0+4}{2}, \frac{4+4}{2} \right) = (2,4)$$

$$H = \left( \frac{0+0}{2}, \frac{0+4}{2} \right) = (0,2)$$

The center of the square will be at (2,2). Label the center point X. The center of the square is also the center of the octagon. Label the vertices of the octagon J, K, L, M, N, O, P, Q.



2. To determine the vertices, the equations of each of the lines can be determined. The slopes are determined using the graph.

The student is expected to:

(C) derive and use formulas involving length, slope, and midpoint.

**(G.8) Congruence and the geometry of size.**

The student uses tools to determine measurements of geometric figures and extends measurement concepts to find area, perimeter, and volume in problem situations.

The student is expected to:

(A) find areas of regular polygons, circles, and composite figures;

**Connections to TAKS:**

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 4: The student will formulate and use linear equations and inequalities.

Objective 7: The student will demonstrate an understanding of two-and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

$\overleftrightarrow{AG}$  :

Slope: -2 Point A(0,4)

$$y - 4 = -2(x - 0)$$

$$y = -2x + 4$$

 $\overleftrightarrow{AF}$ :Slope:  $-\frac{1}{2}$  Point A(0,4)

$$y = -\frac{1}{2}x + 4$$

 $\overleftrightarrow{BG}$  :

Slope: 2 Point B(4,4)

$$y - 4 = 2(x - 4)$$

$$y = 2x - 4$$

 $\overleftrightarrow{BH}$ :Slope:  $\frac{1}{2}$  Point B(4,4)

$$y - 4 = \frac{1}{2}(x - 4)$$

$$y = \frac{1}{2}x + 2$$

 $\overleftrightarrow{CH}$  :Slope:  $-\frac{1}{2}$  Point H(2,0)

$$y - 2 = -\frac{1}{2}(x - 0)$$

$$y = -\frac{1}{2}x + 2$$

$\overleftrightarrow{CE}$ :

Slope: -2    Point E(2,4)

$$y - 4 = -2(x - 2)$$

$$y = -2x + 8$$

 $\overleftrightarrow{DF}$ :Slope:  $\frac{1}{2}$     Point D(0,0)

$$y - 0 = \frac{1}{2}(x - 0)$$

$$y = \frac{1}{2}x$$

 $\overleftrightarrow{DE}$ :

Slope: 2    Point D(0,0)

$$y - 0 = 2(x - 0)$$

$$y = 2x$$

Using linear combination to solve the systems, the values of  $x$  and  $y$  can be found to determine the intersection of lines  $\overleftrightarrow{HC}$  and  $\overleftrightarrow{DF}$ . The point of intersection will be labeled point J as in the diagram above.

$$y = -\frac{1}{2}x + 2$$

$$y = \frac{1}{2}x$$

Adding these 2 equations together produces the following:

$$2y = 2, \text{ therefore } y = 1.$$

Substitute  $y = 1$  into either equation to solve for  $x$ :

$$1 = \frac{1}{2}x$$

$$2 = x$$

Therefore the point of intersection is J(2,1).

The intersection of line  $\overleftrightarrow{AG}$  and  $\overleftrightarrow{HC}$  is the point K.

Using linear combination on the equation for lines  $\overleftrightarrow{AG}$  and  $\overleftrightarrow{HC}$  will allow the coordinates of point K to be found.

$$y = -2x + 4$$

$$y = -\frac{1}{2}x + 2$$

$$0 = -\frac{3}{2}x + 2$$

$$-2 = -\frac{3}{2}x$$

$$\frac{4}{3} = x$$

Substitute  $x = \frac{4}{3}$  into either equation to find the coordinate of y.

$$y = -2\left(\frac{4}{3}\right) + 4$$

$$y = \frac{4}{3}$$

The coordinates of K are  $\left(\frac{4}{3}, \frac{4}{3}\right)$ .

Point L is the intersection of lines  $\overleftrightarrow{AG}$  and  $\overleftrightarrow{DE}$ .

$$\overleftrightarrow{AG}: y = -2x + 4$$

$$\overleftrightarrow{DE}: y = 2x$$

$$2x = -2x + 4$$

$$4x = 4$$

$$x = 1$$

$$y = 2x = 2(1) = 2$$

The coordinates of L are (1,2).

Point M is the intersection of lines  $\overleftrightarrow{DE}$  and  $\overleftrightarrow{BH}$ .

$$\overleftrightarrow{DE}: y = 2x$$

$$\overleftrightarrow{BH}: y = \frac{1}{2}x + 2$$

$$2x = \frac{1}{2}x + 2$$

$$4x = x + 4$$

$$x = \frac{4}{3}$$

$$y = 2x = 2\left(\frac{4}{3}\right) = \frac{8}{3}$$

The coordinates of point M are  $\left(\frac{4}{3}, \frac{8}{3}\right)$ .

Point N is the intersection of lines  $\overleftrightarrow{AF}$  and  $\overleftrightarrow{BH}$ .

$$\overleftrightarrow{AF}: y = -\frac{1}{2}x + 4$$

$$\overleftrightarrow{BH}: y = \frac{1}{2}x + 2$$

$$-\frac{1}{2}x + 4 = \frac{1}{2}x + 2$$

$$-1x = -2$$

$$x = 2$$

$$y = \frac{1}{2}x + 2 = 1 + 2 = 3$$

The coordinates of N are (2,3).

Point O is the intersection of lines  $\overleftrightarrow{AF}$  and  $\overleftrightarrow{CE}$ .

$$\overleftrightarrow{AF}: y = -\frac{1}{2}x + 4$$

$$\overleftrightarrow{CE}: y = -2x + 8$$

$$-\frac{1}{2}x + 4 = -2x + 8$$

$$-1x + 8 = -4x + 16$$

$$x = \frac{8}{3}$$

$$y = -2x + 8 = -2\left(\frac{8}{3}\right) + 8 = \frac{8}{3}$$

The coordinates of O are  $\left(\frac{8}{3}, \frac{8}{3}\right)$ .

Point P is the intersection of lines  $\overleftrightarrow{CE}$  and  $\overleftrightarrow{BG}$ .

$$\overleftrightarrow{CE}: y = -2x + 8$$

$$\overleftrightarrow{BG}: y = 2x - 4$$

$$2x - 4 = -2x + 8$$

$$4x = 12$$

$$x = 3$$

$$y = 2x - 4 = 2(3) - 4 = 2$$

The coordinates of P are (3,2).



Point Q is the intersection of lines  $\overleftrightarrow{DF}$  and  $\overleftrightarrow{BG}$ .

$$\overleftrightarrow{DF}: y = \frac{1}{2}x$$

$$\overleftrightarrow{BG}: y = 2x - 4$$

$$\frac{1}{2}x = 2x - 4$$

$$x = 4x - 8$$

$$-3x = -8$$

$$x = \frac{8}{3}$$

$$y = \frac{1}{2}x = \frac{1}{2}\left(\frac{8}{3}\right) = \frac{4}{3}$$

The coordinates of Q are  $\left(\frac{8}{3}, \frac{4}{3}\right)$ .

3. A regular octagon has 8 congruent sides and 8 congruent angles.

It can be demonstrated that the octagon is not a regular octagon by determining  $LK$ ,  $KJ$ ,  $XK$ ,  $XJ$ , and  $XL$ .

$$LK = \sqrt{\left(1 - \frac{4}{3}\right)^2 + \left(2 - \frac{4}{3}\right)^2} = \sqrt{\left(-\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = \sqrt{\frac{5}{9}}$$

$$KJ = \sqrt{\left(2 - \frac{4}{3}\right)^2 + \left(1 - \frac{4}{3}\right)^2} = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2} = \sqrt{\frac{5}{9}}$$

$$XK = \sqrt{\left(2 - \frac{4}{3}\right)^2 + \left(2 - \frac{4}{3}\right)^2} = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = \sqrt{\frac{8}{9}}$$

$$XL = \sqrt{(2-1)^2 + (2-2)^2} = 1$$

$$XJ = \sqrt{(2-2)^2 + (2-1)^2} = 1$$

$\overline{XK}$  is not equal in length to  $\overline{XL}$  or  $\overline{XJ}$ .

Triangles  $XKJ$  and  $XLK$  are congruent triangles, but they are not isosceles triangles because the sides are of three different lengths.

$$XL = XJ = 1$$

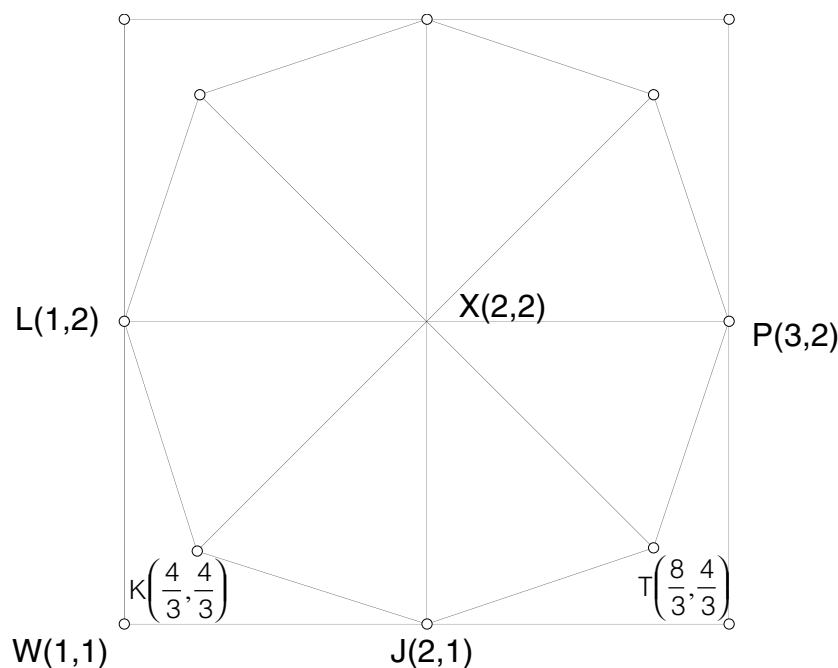
$$LK = KJ = \sqrt{\frac{5}{9}}$$

The length of the common side  $\overline{XK}$  is  $\sqrt{\frac{8}{9}}$ .

Thus, the octagon is not a regular octagon.

4. It is still possible to find the area of the octagon as it can be divided into 4 parts that are congruent. Each part of the bottom half is composed of the two congruent triangles.

For example, one-fourth of the octagon,  $LKJX$ , is composed of the two congruent triangles  $LXK$  and  $JXK$ .



The area of square LXJW is 1 square unit.

Draw  $\overline{WK}$ . Triangle WJK has a base,  $\overline{WJ}$ , that measures one unit.

The height from K to  $\overline{WJ}$  is  $\frac{4}{3} - 1 = \frac{1}{3}$ .

The area of the triangle is  $\frac{1}{2} \cdot 1 \cdot \frac{1}{3} = \frac{1}{6}$ .

Similarly, the area of triangle LWK is  $\frac{1}{2} \cdot 1 \cdot \frac{1}{3} = \frac{1}{6}$ .

The area of LKJX is the area of LXJW minus the area of the two triangles LWK and WKJ, or

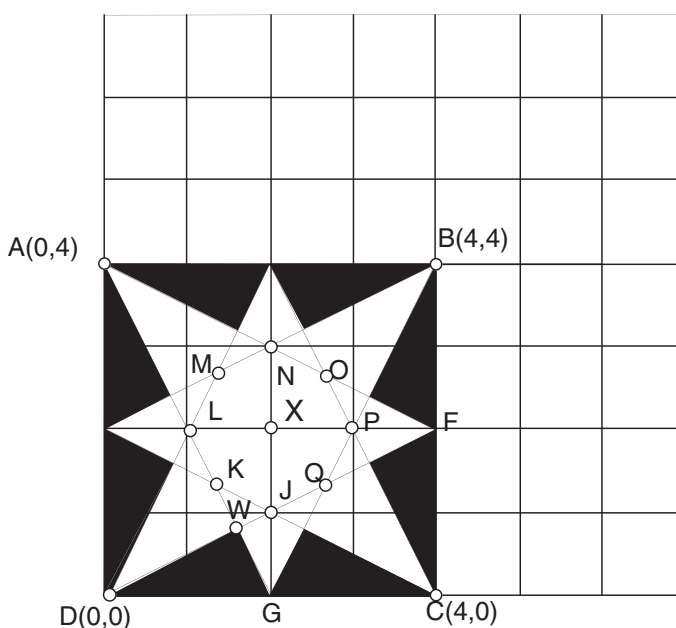
$$1 - \frac{1}{6} - \frac{1}{6} = \frac{2}{3}.$$

The area of the octagon is four times the area of LKJX, or  $4\left(\frac{2}{3}\right) = \frac{8}{3} = 2\frac{2}{3}$  square inches.

Ed's design meets the guidelines because the total area is no larger than 3 square inches.

**Extension Question:**

- Determine the area of the white star.



Call the intersection of the lines  $\overleftrightarrow{DF}$  and  $\overleftrightarrow{AG}$  point  $W$ .

$$\overleftrightarrow{DF}: \quad y = \frac{1}{2}x$$

$$\overleftrightarrow{AG}: \quad y = -2x + 4$$

$$\frac{1}{2}x = -2x + 4$$

$$x = -4x + 4$$

$$x = \frac{8}{5}$$

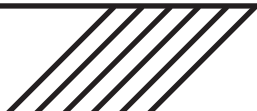
$$y = \frac{1}{2}x = \frac{1}{2}\left(\frac{8}{5}\right) = \frac{4}{5}$$

Triangle  $DWG$  has a base of 2 inches and a height  $\frac{4}{5}$  inches.

The area of triangle  $DWG$  is  $\frac{1}{2} \cdot 2 \cdot \frac{4}{5} = \frac{4}{5}$  square inches.

The area of the star is the area of the large square minus 8 of the triangles that are congruent to triangle  $DWG$ .

$$16 - 8\left(\frac{4}{5}\right) = 16 - \frac{32}{5} = 16 - 6\frac{2}{5} = 9\frac{3}{5} \text{ square inches.}$$



**Student Work Sample**

The student work displayed on the next page was completed using geometry software.

The criteria of the Geometry Solution Guide exemplified here are the following:

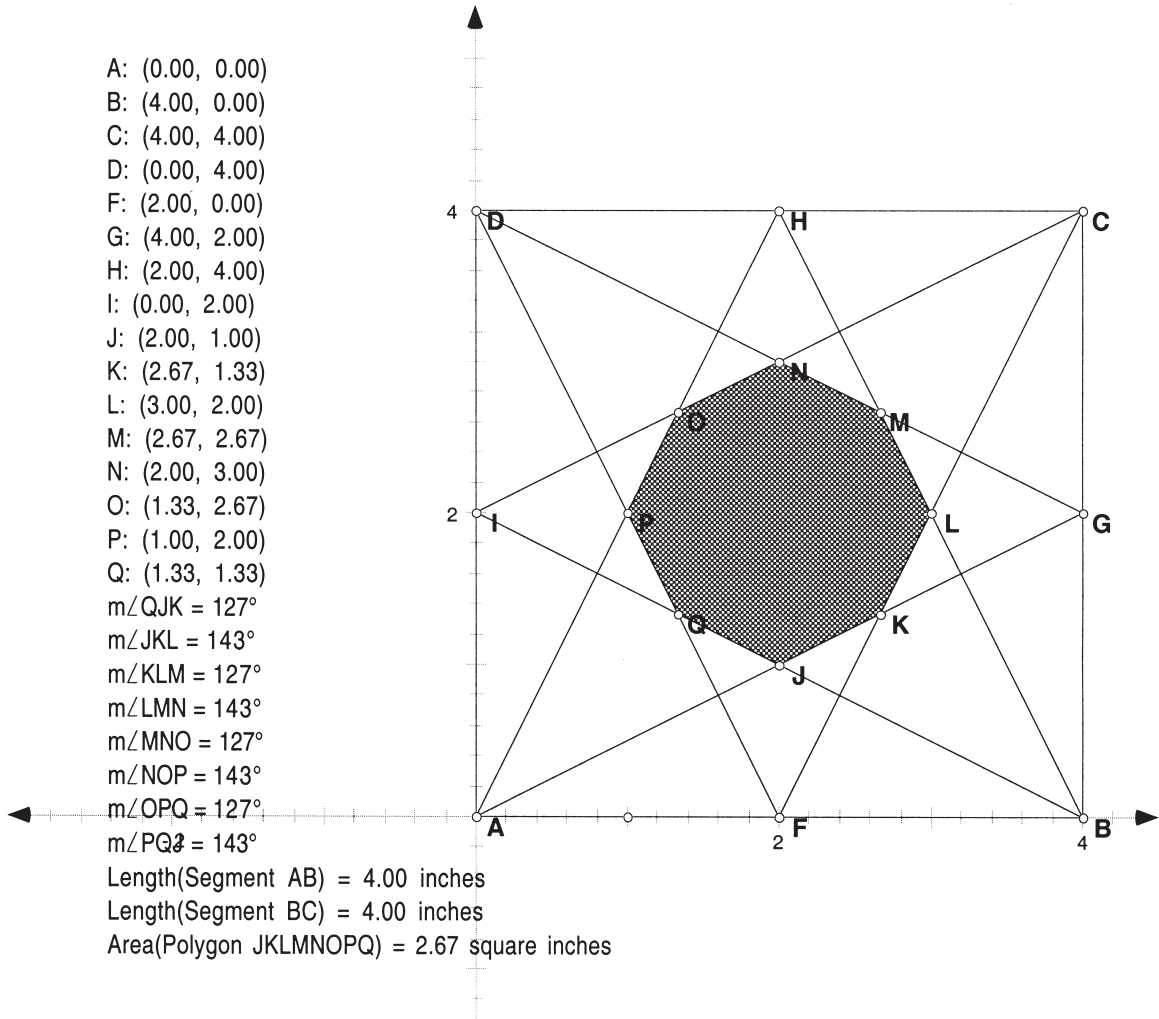
- Makes an appropriate and accurate representation of the problem using correctly labeled diagrams.

*The student has clearly and correctly labeled points. The measurements have been taken that will allow the student to answer the questions about the quilt design.*

- Uses appropriate tools.

*The task did not require that the measurements or calculations be done using algebraic methods. The use of the geometric software demonstrates the solution, but note that it does constitute a geometric proof.*

These are my coordinates for each design point. From each point to another point the points are listed. The coordinates are accurate on the graph.



The octagon is not a regular octagon. The angles are not the same.

This quilt matches the guidelines to enter the contest. The quilt is a 4 in quilt and the shape is not larger than 3 square inches.





## Wearable Art

Lorraine's graphics arts class has been assigned a t-shirt design project. Each student is to create a design by drawing any quadrilateral, connecting the midpoints of the sides to form another quadrilateral, and coloring the regions. Lorraine claims that everyone's inner quadrilateral will be a parallelogram. Use coordinate geometry to determine if she is correct. Show all of your work, and explain your reasoning.

**Notes**

**Materials:**

One graphing calculator per student

Computer drawing utility (optional)

**Geometry TEKS Focus:**

(G.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student is expected to:

(B) make conjectures about angles, lines, polygons, circles, and three-dimensional figures and determine the validity of the conjectures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(G.7) **Dimensionality and the geometry of location.**

The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student is expected to:

(B) use slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons; and

(C) derive and use formulas involving length, slope, and midpoint.

**Teacher Notes**

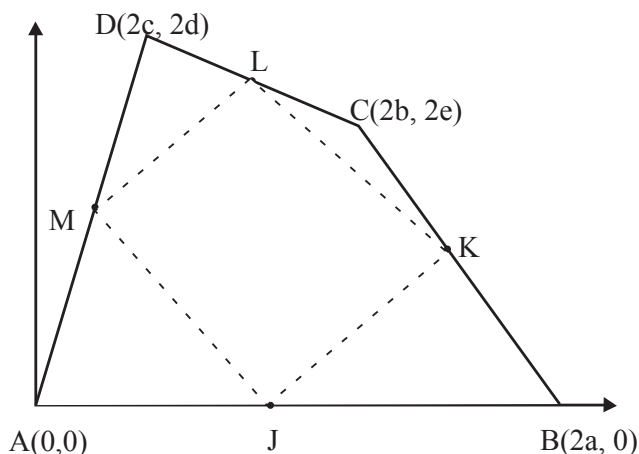
This performance task addresses the same mathematical concepts as the problem in *Whitebeard’s Treasure* that gives the numerical coordinates. In this task, the coordinates are given and the student must represent the situation using variable coordinates for the points. The teacher may choose to use one or both of these tasks.

**Scaffolding Questions:**

- How will using randomly selected coordinates for the vertices of the quadrilateral help prove Lorraine’s claim?
- How could you place your quadrilateral on the coordinate plane so that the coordinates of your vertices will be “easy” to work with?
- How can you select coordinate values that will be “friendly” when figuring midpoints?

**Sample Solutions:**

Draw a quadrilateral on a coordinate plane. Locate one vertex at the origin and one side on the x-axis. Remembering that midpoints will be needed, select coordinates that are multiples of two. Label the vertices of the quadrilateral A, B, C, and D and the midpoints J, K, L, and M.



Find the coordinates of the midpoints as follows:

The midpoint of  $\overline{AB}$ , J, is the point

$$\left(\frac{0+2a}{2}, \frac{0+0}{2}\right) = (a, 0)$$

The midpoint of  $\overline{BC}$ , K, is the point

$$\left(\frac{2a+2b}{2}, \frac{0+2e}{2}\right) = (a+b, e)$$

The midpoint of  $\overline{CD}$ , L, is the point

$$\left(\frac{2b+2c}{2}, \frac{2d+2e}{2}\right) = (b+c, d+e)$$

The midpoint of  $\overline{DA}$ , M, is the point

$$\left(\frac{2c+0}{2}, \frac{2d+0}{2}\right) = (c, d)$$

Draw line segments  $\overline{JK}$ ,  $\overline{KL}$ ,  $\overline{LM}$ , and  $\overline{MJ}$ . These segments form the sides of the inner quadrilateral.

If the inner quadrilateral is a parallelogram, then opposite sides must be parallel. Using the slope formula, it can be shown that the lines are parallel because parallel lines have the same slope.  $\overline{JK}$  must be parallel to  $\overline{LM}$ , and  $\overline{KL}$  must be parallel to  $\overline{MJ}$  if the figure is a parallelogram.

The slope of  $\overline{JK} = \frac{e-0}{(a+b)-a} = \frac{e}{b}$ .

The slope of  $\overline{LM} = \frac{(d+e)-d}{(b+c)-c} = \frac{e}{b}$ .

Segments  $\overline{JK}$  and  $\overline{LM}$  both have the same slope, therefore they are parallel.

The slope of  $\overline{MJ} = \frac{d-0}{c-a} = \frac{d}{c-a}$ .

**Additional Geometry TEKS:**

(G.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student is expected to:

(A) use constructions to explore attributes of geometric figures and to make conjectures about geometric relationships; and

(G.7) **Dimensionality and the geometry of location.**

The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student is expected to:

(A) use one- and two-dimensional coordinate systems to represent points, lines, rays, line segments, and figures;

**Connections to TAKS:**

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

The slope of  $\overline{KL} = \frac{(d+e)-e}{(b+c)-(a+b)} = \frac{d}{c-a}$ .

Segments  $\overline{MJ}$  and  $\overline{KL}$  both have the same slope. Therefore, they are parallel.

By definition, quadrilateral JKLM is a parallelogram (both pairs of opposite sides are parallel). Lorraine's claim will be true for any quadrilateral that joins the midpoints of the sides of any convex quadrilateral.

**Extension Questions:**

- Mark claims that the inner quadrilateral created by joining the midpoints of the sides will be a rhombus. Prove or disprove his conjecture. Be sure to show all of your work.

A drawing utility can be used to test the conjecture. A rhombus has perpendicular diagonals and four congruent sides. In this case  $\overline{MK}$  is not perpendicular to  $\overline{JL}$  because the slopes of the diagonals are not opposite reciprocals of one another.

The slope of  $\overline{MK}$  is  $\frac{e-d}{a+b-c}$ .

The slope of  $\overline{JL}$  is  $\frac{d+e-0}{b+c-a}$ .

$$-\frac{1}{\frac{d+e-0}{b+c-a}} = -\frac{b+c-a}{d+e-0} \neq \frac{e-d}{a+b-c}$$

The slopes of the two lines are not opposite reciprocals. Therefore, the quadrilateral is not a rhombus.

There is no need to check the four congruent sides because the first condition of a rhombus was not met.

- Can the quadrilateral formed by joining the midpoints be a rectangle? A square? Write your conjectures and test them using a drawing program. Justify your answers.

If the quadrilateral is a rectangle, then  $\overline{MJ}$  must be perpendicular to  $\overline{JK}$ . That would mean that the slope of  $\overline{MJ}$  is equal to the opposite reciprocal of the slope of  $\overline{JK}$ . Recall that the slope of  $\overline{MJ} = \frac{d}{c-a}$  and the slope of  $\overline{JK} = \frac{e}{b}$ .

$$\frac{d}{c-a} = \frac{-1}{\frac{e}{b}}$$

$$\frac{d}{c-a} = \frac{-b}{e}$$

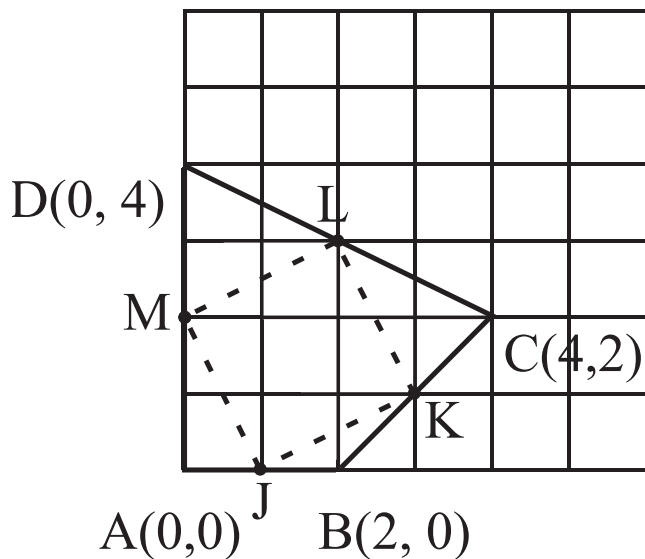
$$de = -bc + ab$$

Choose any values for  $a, b, c, d,$  and  $e$  that make  $de = -bc + ab$  a true statement. One set of values is  $d = 2, e = 1, a = 1, c = 0, b = 2.$

$$de = -bc + ab$$

$$2(1) = -2(0) + 1(2)$$

This choice results in a quadrilateral with vertices  $A(0,0), B(2,0), C(4,2),$  and  $D(0,4).$  Connecting the midpoints of each side forms a rectangle, as shown below.



The midpoints are  $J(1,0); K(3,1); L(2,3);$  and  $M(0,2).$

Each side of the rectangle is the hypotenuse of a right triangle with legs measuring 1 unit and 2 units. Therefore, by the Pythagorean Theorem, the length of the hypotenuse is

$$\sqrt{1^2 + 2^2} = \sqrt{5}$$

The resulting figure, JKLM, is also a square because it is a rectangle with four congruent sides.



## Sea Quest

The Coastal Marine Association is going to gather data in Galveston Bay. A new computer device will be submerged just outside the ship channel fairway. In order to get the most accurate readings, the device must be located at a point such that the distance from the bay floor to the device is equal to one-third of the distance between the device and the top of the antenna. The depth of the bay at the selected location is 18.5 feet. The height of the antenna will be 21.5 feet above the water at mean tide.

Draw a diagram of the situation, and determine the depth at which the new computer device will be located at mean tide. Justify your solution.

## Notes

### Materials:

One graphing calculator per student

### Geometry TEKS Focus:

(G.7) **Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student is expected to:

(A) use one- and two-dimensional coordinate systems to represent points, lines, rays, line segments, and figures;

### (G.5) Geometric patterns.

The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to:

(A) Use numeric and geometric patterns to develop algebraic expressions representing geometric properties;

## Teacher Notes

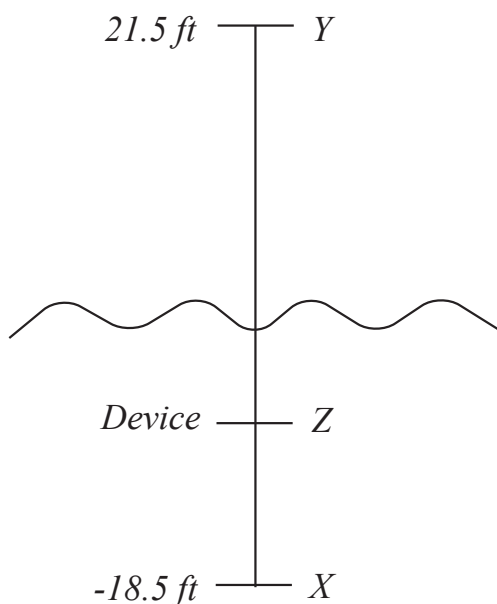
### Scaffolding Questions:

- If you assign numerical values to the objects, where will the zero value be?
- What type of diagram would best help illustrate this situation?
- How can you find the total distance between the floor of the bay and the top of the antenna?

### Sample Solutions:

If the depth of the water is 18.5 feet, and the height of the antenna is 21.5 feet, both positive and negative numbers will need to be used. The depth will correspond to -18.5 feet because the distance is below the water. The height of the antenna will correspond to a positive 21.5 feet.

A vertical number line can be used to illustrate this situation.



The distance between X and Z is equal to  $\frac{1}{3}$  the distance between Z and Y. Using segment addition, it can be shown that  $XZ + ZY = XY$ , and we are told that  $XZ = \frac{1}{3}(ZY)$ . The distance between X and Y is  $|-18.5 - 21.5|$  or 40 feet.



To find the values of  $\overline{XZ}$  and  $\overline{ZY}$ , the system of two equations may be solved by substitution.

$$\begin{cases} XZ + ZY = XY \\ XZ = \frac{1}{3}(ZY) \end{cases}$$

Using substitution,

$$\frac{1}{3}(ZY) + ZY = 40$$

$$\frac{4}{3}(ZY) = 40$$

$$\frac{3}{4} \cdot \frac{4}{3}(ZY) = \frac{3}{4} \cdot 40$$

$$ZY = 30$$

Therefore,  $XZ = 10$  because  $XZ + ZY = 40$ .

The depth at point Z must be 10 feet from the floor of Galveston Bay. If the depth at the selected location is 18.5 feet, the computer device will be located at a depth of 8.5 feet. ( $-18.5 + 10 = -8.5$ ).

**Connections to TAKS:**

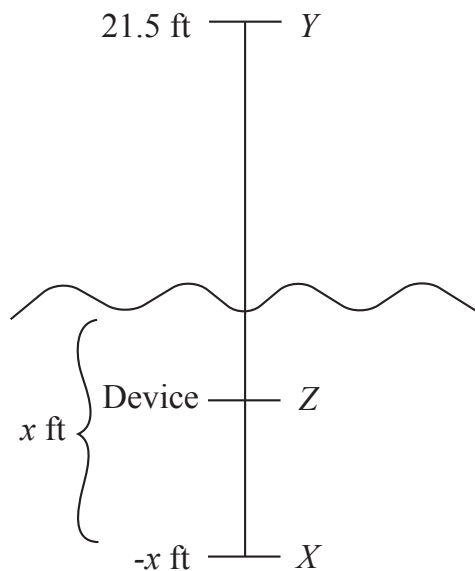
Objective 4: The student will formulate and use linear equations and inequalities.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

**Extension Question:**

- How deep would the bay have to be in order to locate the computer device at half the depth of the bay? Assume the device still gives the most accurate readings when the distance from the bay floor to the device is one-third the distance from the device to the top of the antenna, 21.5 feet above the water.

Let  $x$  represent the depth of the bay. The coordinate  $-x$  would represent the depth. The location of the device would be represented by  $-\frac{1}{2}x$ .



The distance from the top of the antenna to the device would be presented by  $21.5 - \left(-\frac{1}{2}x\right)$ .

Half of the depth is  $\frac{1}{2}x$  which must be equal to one-third of the distance from the top of the antenna to the device.

$$\frac{1}{2}x = \frac{1}{3}\left(21.5 - \left(-\frac{1}{2}x\right)\right)$$

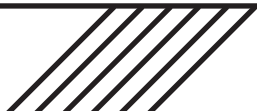
$$\frac{1}{2}x = \frac{21.5}{3} + \frac{1}{6}x$$

$$3x = 43 + x$$

$$2x = 43$$

$$x = 21.5$$

The depth of the bay would have to be 21.5 feet.





## **Chapter 2:**

*Patterns,  
Conjecture,  
and Proof*



# Introduction

The assessments in Chapter 2 emphasize geometric thinking and spatial reasoning. High school geometry affords one of the first opportunities for students to explore the structure of mathematical systems and to understand what it means to prove mathematically that a conjecture is true. The emphasis of the performance tasks in this chapter is on discovering this structure through investigations, observing patterns, and formulating conjectures based on these experiences. Once the conjecture is formed, the students are asked to justify why it is or is not true.

The Mad as a Hatter or Hat as a Madder activity assesses the students understanding of logical reasoning. The rest of the tasks are grouped together to emphasize the process of having students investigate, make conjectures, and then justify their conjectures. For example, the Pizza Delivery Service Regions activity provides the opportunity for investigation and conjecture while More Pizza Delivery requires the verification of students' conjectures.





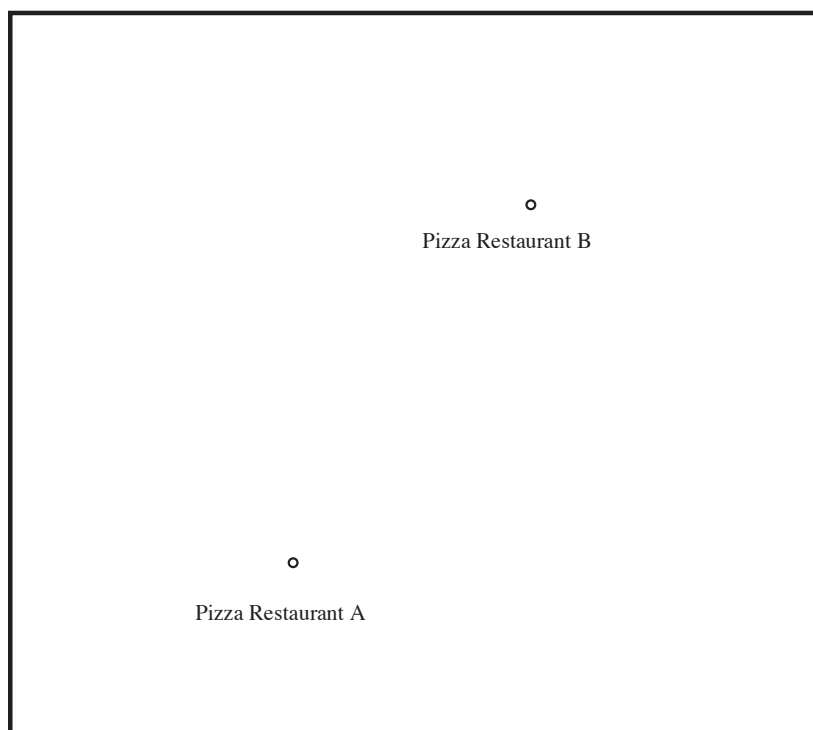
## Pizza Delivery Service Regions

### Problem 1

#### *Two Restaurants*

The rectangle below represents a map of a city, and the two points represent pizza restaurants. Your task is to accurately determine the delivery service region for each of the restaurants. A household is in a restaurant's delivery service region if it is located closer to that restaurant than to the other restaurant.

You will define each restaurant's delivery service region by using compass and straightedge to accurately construct the boundary between the two regions.



- a) Write a few sentences explaining how you determined where to locate the boundary.

b) Based on your observations and construction, what do you think must be true about a household located on the boundary between service regions?

Complete the following conjectures: If a household is on the boundary between two service regions, then the household \_\_\_\_\_

\_\_\_\_\_.

If a point is on a line segment's perpendicular bisector, then the point \_\_\_\_\_

\_\_\_\_\_.

\_\_\_\_\_.

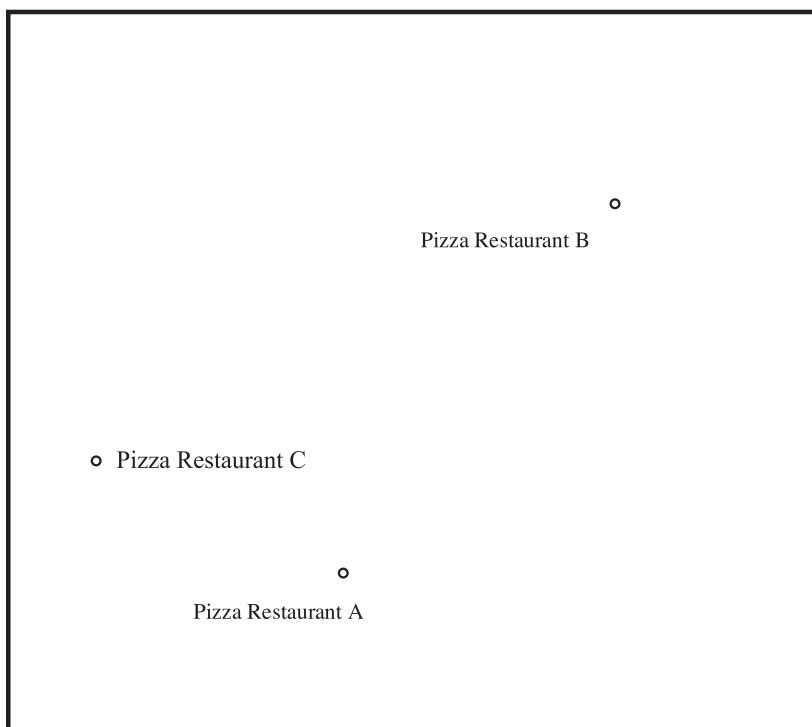
## Problem 2

### *Three Restaurants*

Determine the delivery service regions for pizza restaurants A, B, and C.

As in the previous task, a household is located in a delivery service region if it is closer to that restaurant than to either of the other two restaurants.

As before, you can define each restaurant's service region by using compass and straightedge to accurately construct the boundaries between each of the three regions.



- a) Based on your observations and construction, what do you think must be true about a household located on all the boundaries between three service regions?

Complete the following conjectures: If a household is on all the boundaries between three service regions, then the household

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The perpendicular bisectors of the three segments that form a triangle intersect at a point which is \_\_\_\_\_

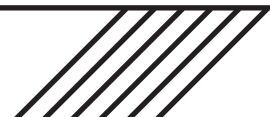
\_\_\_\_\_

\_\_\_\_\_.

**Problem 3**

*Your Conjecture*

Using your conjecture from Problem 1, write a few sentences explaining why your conjecture from Problem 2 must be true. In other words, if your conjecture about households on the boundary between two service regions is true, then use this fact to explain why your conjecture about households on the boundary between three service regions must, therefore, be true.





## Notes

### Materials:

One compass and straightedge per student

### Geometry TEKS Focus:

(G.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student is expected to:

(A) use constructions to explore attributes of geometric figures and to make conjectures about geometric relationships; and

### Additional Geometry TEKS:

(G.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student is expected to:

(B) make conjectures about angles, lines, polygons, circles, and three-dimensional figures and determine the validity of the conjectures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(G.3) **Geometric structure.** The student applies logical reasoning to justify and prove mathematical statements.

The student is expected to:

(D) use inductive reasoning to formulate a conjecture; and

### Connections to TAKS:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 7: The student will demonstrate an understanding

## Teacher Notes

This performance task is intended to determine if a student can make conjectures based on what they observe (use inductive reasoning). The next task, More Pizza Delivery, will determine if the students can use deductive reasoning to prove the statement.

### Student Response to Problem 1, Question 2:

“It means you can order service from either one, because the two are equal distance from you. I myself would buy pizza from the one that delivers faster and has better pizza.”

### Scaffolding Questions:

#### Problem 1

#### Two Restaurants

- If you mark a point at random on the map to represent a household, how can you use your compass to determine which restaurant is closer to the household?
- If you mark a point at random on the map to represent a household, how can you use your compass to determine the locations of other households at that same distance from either of the two restaurants?
- Use your compass to locate at least three households that are the same distance from both restaurants. Is it possible to find all the households that are the same distance from both restaurants?
- If you have not already done so, construct the perpendicular bisector of the line segment joining restaurant A and restaurant B. What appears to be true about all the points on the perpendicular bisector of the line segment joining restaurant A and restaurant B?

*Problem 2*

## Three Restaurants

- Which construction could you use to draw the boundary between the service regions of restaurant A and restaurant C?
- Which construction could you use to draw the boundary between the service regions of restaurant C and restaurant B?
- What appears to be true about the three boundary lines you have constructed?

*Problem 3*

## Your Conjecture

- What do you think is true about a household on the boundary line between restaurants A and B?
- What appears to be true about a point located on the perpendicular bisector of the segment joining restaurants A and B?
- What do you think is true about a household on the boundary line between restaurants A and C?
- What seems to be true about a point located on the perpendicular bisector of the segment joining restaurants A and C?
- If your conjecture is proven to be true, what must be true about a household located on both boundary lines?
- If your conjecture is proven to be true, what must be true about a household located on both perpendicular bisectors?

**Sample Solutions:***Problem 1*

- a) Construct the perpendicular bisector of the segment joining restaurant A and restaurant B. The compass setting represents the distance between a household and a restaurant. A circle with its center at one of

of two-and three-dimensional representations of geometric relationships and shapes.

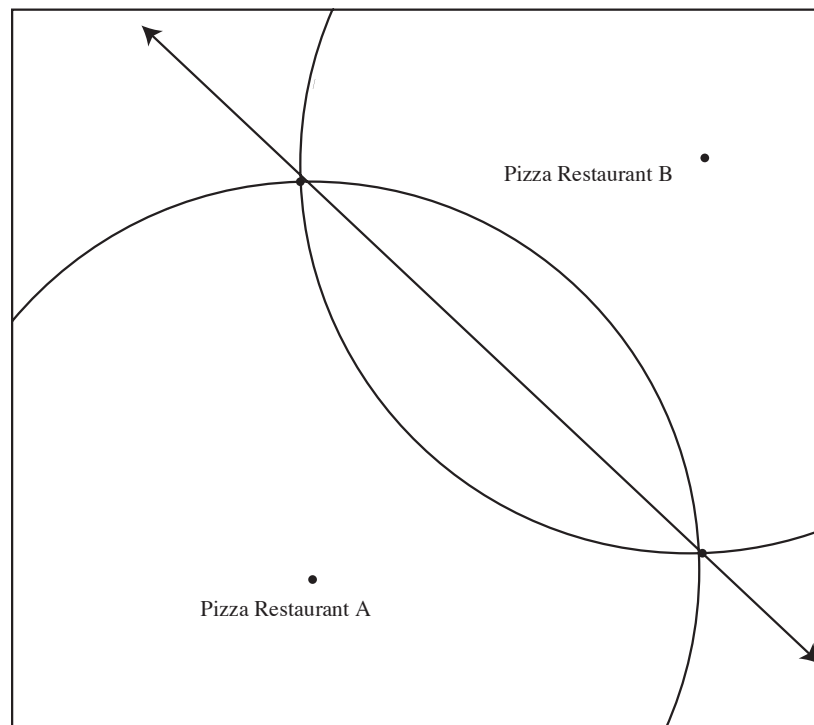
**One teacher says . . .**

“To introduce this problem I explained they had all just been hired to work for the pizza company to decide on delivery areas and to be cost effective.”

**One student says in response to Problem 1, Question 2 . . .**

“It means you can order service from either one, because the two are equal distance from you. I myself would buy pizza from the one that delivers faster and has better pizza.”

the restaurants, and a fixed compass setting as a radius, represents all households at that fixed distance from the restaurant. A household is the same distance from both restaurants only if it is located on the perpendicular bisector.



- b) Complete the following conjectures: If a household is on the boundary between two service regions, then the household is the same distance from restaurant A as it is from restaurant B.

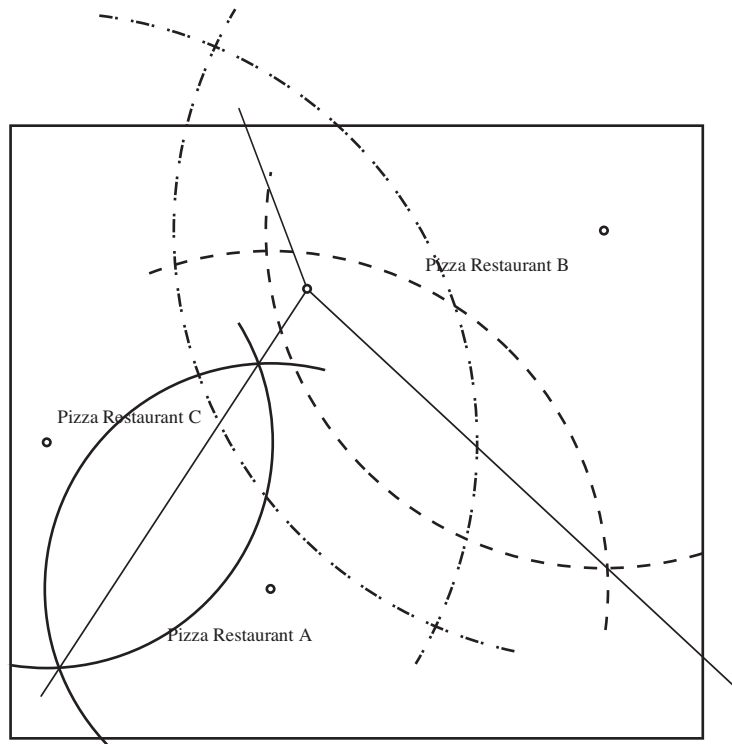
If a point is on a line segment's perpendicular bisector, then the point is the same distance from each of the line segment's endpoints.

### *Problem 2*

Construct the three perpendicular bisectors of the line segments joining restaurants A, B, and C. Repeat the construction from problem 1 to find the service region boundaries between restaurants A and C and between restaurants B and C.

These boundary lines intersect at a single point. This point is the circumcenter of the triangle formed by joining the points representing the three restaurants.





- a) Complete the following conjectures: If a household is on the intersection point that is the boundary between three service regions, then the household is the same distance from all three restaurants.

The perpendicular bisectors of the three segments that form a triangle intersect at a point which is the same distance from all three of the triangles' vertices.

### Problem 3

Note: Students can be required to answer this question using both the terms specific to the problem's context (households, service regions, boundaries) as well as the more formal language of geometric abstraction (points, line segments, perpendicular bisectors).

A household located at the intersection of the three service region boundaries must be the same distance from all three restaurants because of the following:

- This household is on the boundary between restaurant A's and restaurant B's service regions, and because of the conjecture from Problem 1, it is, therefore, the same distance from restaurant A as it is from restaurant B.
- This household is also on the boundary between restaurant A's and restaurant C's service regions (or alternately, restaurant B's and restaurant C's service regions), and because of the conjecture from Problem 1, it is, therefore, the same distance from restaurant A as it is from restaurant C (or alternately, restaurant B from restaurant C).

This makes the household the same distance from all three restaurants.

A point located at the intersection of the perpendicular bisectors of a triangle must be the same distance from all three of the triangle's vertices because of the following:

- This point is on the perpendicular bisector of the line segment joining vertices A and B, and because of the conjecture from Problem 1, it is, therefore, the same distance from vertex A as it is from vertex B.
- This point is also on the perpendicular bisector of the line segment joining vertices A and C (or alternately, vertices B and C), and because of the conjecture from Problem 1, it is, therefore, the same distance from vertex A as it is from vertex C (or alternately, vertex B and vertex C).

**Extension Questions:**

- Suppose that the three restaurants are located on a coordinate grid at the points A (13,17), B (4,11), and C (3,21). Determine the equations of the boundary lines for the service regions and the circumcenter of the triangle ABC.

To determine the equation of the perpendicular bisector of  $\overline{AB}$ , find the midpoint of the segment, and determine the opposite reciprocal of the slope of  $\overline{AB}$ .

The midpoint of  $\overline{AB}$  is  $\left(\frac{13+4}{2}, \frac{17+11}{2}\right)$  or (8.5,14).

The slope of  $\overline{AB}$  is  $\frac{17-11}{13-4} = \frac{6}{9} = \frac{2}{3}$ .

The slope of a perpendicular line is the opposite reciprocal,  $-\frac{3}{2}$ .

The equation of the perpendicular bisector is

$$y - 14 = -\frac{3}{2}(x - 8.5)$$

$$y = -\frac{3}{2}x + 26.75$$

To determine the equation of the perpendicular bisector of  $\overline{AC}$ , find the midpoint of the segment, and determine the opposite reciprocal of the slope of  $\overline{AC}$ .

The midpoint of  $\overline{AC}$  is  $\left(\frac{13+3}{2}, \frac{17+21}{2}\right)$  or (8,19).

The slope of  $\overline{AC}$  is  $\frac{17-21}{13-3} = \frac{-4}{10} = -\frac{2}{5}$ .

The slope of a perpendicular line is the opposite reciprocal,  $\frac{5}{2}$ .

The equation of the perpendicular bisector is

$$y - 19 = \frac{5}{2}(x - 8)$$

$$y = \frac{5}{2}x - 1$$

To determine the equation of the perpendicular bisector of  $\overline{BC}$ , find the midpoint of the segment and determine the opposite reciprocal of the slope of  $\overline{BC}$ .

The midpoint of  $\overline{BC}$  is  $\left(\frac{3+4}{2}, \frac{21+11}{2}\right)$  or  $(3.5, 16)$ .

The slope of  $\overline{BC}$  is  $\frac{21-11}{3-4} = \frac{10}{-1} = -10$ .

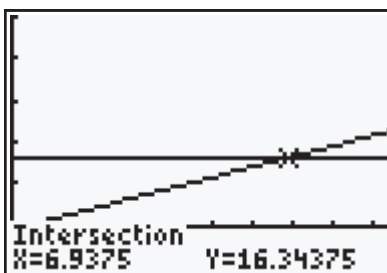
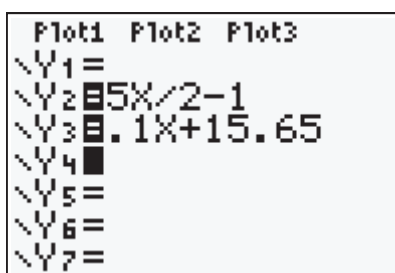
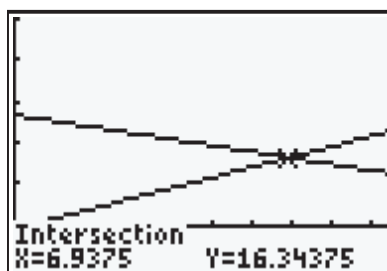
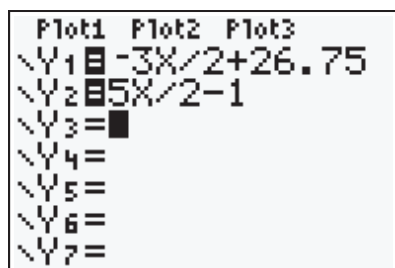
The slope of a perpendicular line is the opposite reciprocal,  $\frac{1}{10}$ .

The equation of the perpendicular bisector is

$$y - 16 = \frac{1}{10}(x - 3.5)$$

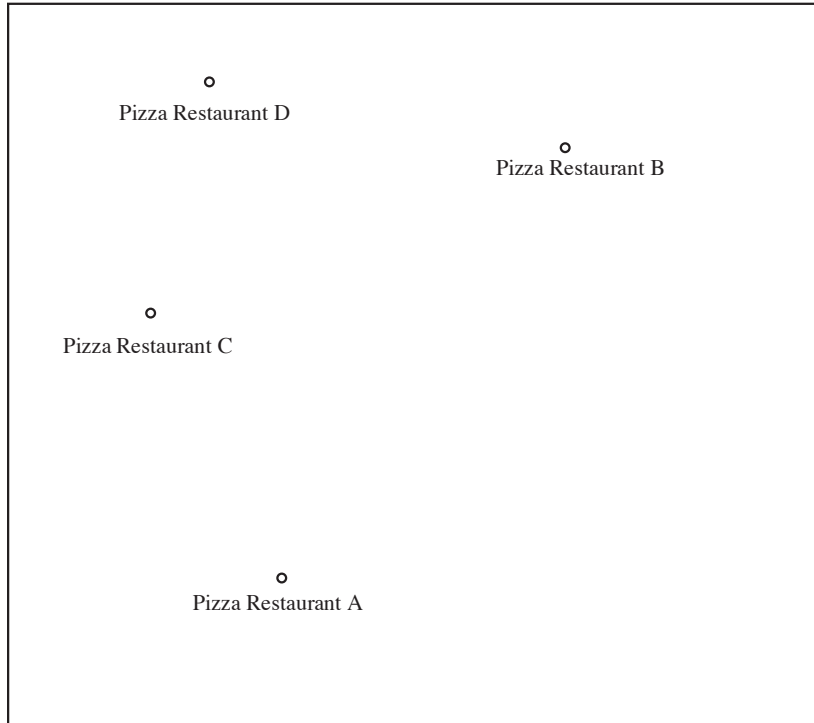
$$y = \frac{1}{10}x + 15.65$$

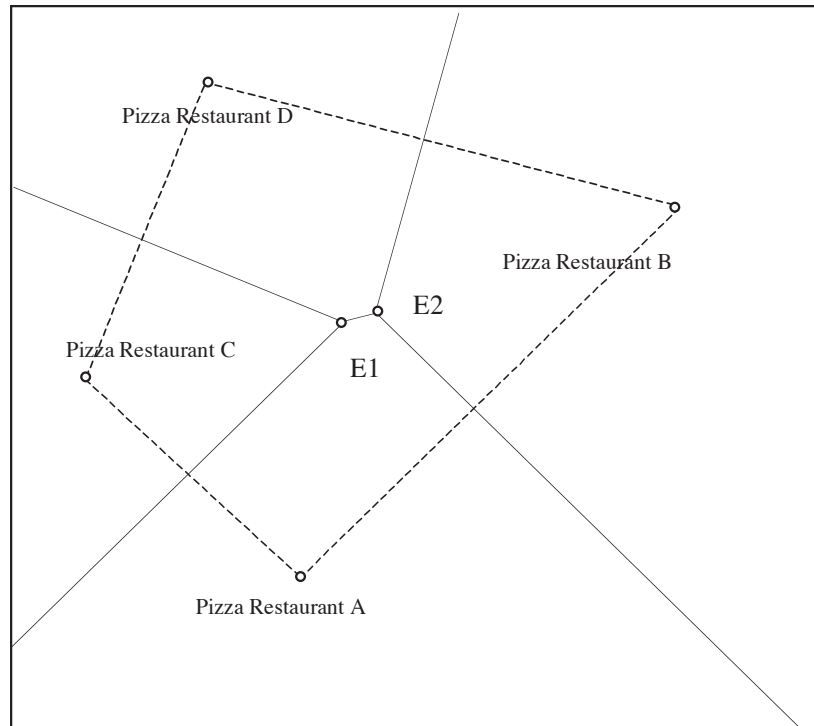
The intersection point of the three lines may be found by using a graphing calculator.



The intersection point is  $(6.9375, 16.34375)$ .

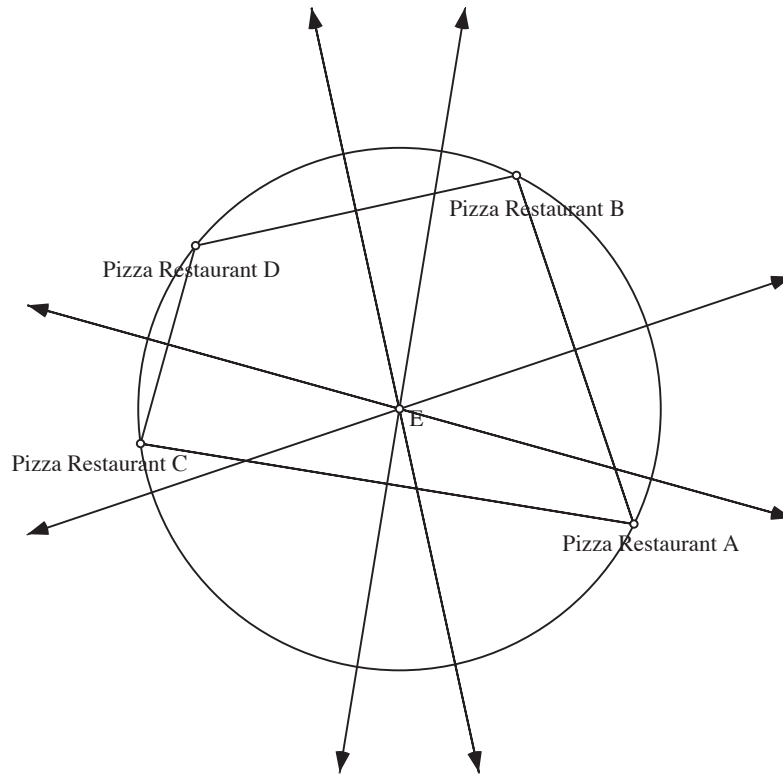
- Determine the pizza service boundary regions for restaurants A, B, C, D.





Point E1 is the circumcenter of the triangle formed by restaurants A, C, and D.  
Point E2 is the circumcenter of the triangle formed by restaurants A, B, and D.

- Position the four restaurants so that there is a single household that is the same distance from all four restaurants. (Challenge: you must accomplish this task without positioning the restaurants so that they form the vertices of a square or a rectangle.)



The four restaurants must be located on a circle. The household represented by point E is equidistant from all four restaurants since it is the center of the circle. The four restaurants must be the vertices of a cyclic quadrilateral.

## More Pizza Delivery

In the task about pizza delivery service regions, you used constructions to define the boundary between the delivery regions for two pizza restaurants.

You then used your construction to make the following conjecture:

If a point is on a line segment's perpendicular bisector, then the point is the same distance from each of the line segment's endpoints.

### Problem 1

Choose one of the following methods to verify the conjecture:

#### *Axiomatic Approach*

Using the definitions, postulates, and theorems of your geometry course, write a deductive proof of the conjecture.

#### *Coordinate Geometry Approach*

Given pizza restaurant A, with coordinates  $A(2,-7)$ , and pizza restaurant B, with coordinates  $B(8,11)$ :

- Find the equation of the perpendicular bisector of  $\overline{AB}$ .
- Use the equation you found to find the coordinates of a point on the perpendicular bisector of  $\overline{AB}$ .
- Verify that the point you found satisfies the conditions of the conjecture.

### Problem 2

#### *Transformational Approach*

Use transformational geometry to verify the converse of the conjecture:

If a point is equidistant from the two endpoints of a line segment, then the point is on the segment's perpendicular bisector.

Given pizza restaurant A, with coordinates  $A(2,-7)$ , and pizza restaurant

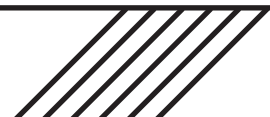
B, with coordinates  $B(8,11)$ , and given point D, with coordinates  $D(8,1)$ , and point E, with coordinates  $E(2,3)$ :

- a) Use coordinate geometry formulas to verify:  $AD = BD$  and  $AE = BE$ .
- b) Find the equation of the line containing points D and E.
- c) Use transformations to show that this line is the perpendicular bisector of  $\overline{AB}$ .

Recall that a reflection takes a pre-image point and moves it across a *mirror line*, so that the *mirror line* is the perpendicular bisector of the segment connecting the point and its image.

Verify that point B is the reflection image of point A across line  $\overleftrightarrow{DE}$ . Thus,  $\overleftrightarrow{DE}$  is the perpendicular bisector of  $\overline{AB}$ .







## Notes

### Materials:

Graph paper

### Geometry TEKS Focus:

(G.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student is expected to:

(B) make conjectures about angles, lines, polygons, circles, and three-dimensional figures and determine the validity of approaches such as coordinate, transformational, or axiomatic.

### (G.7) Dimensionality and the geometry of location.

The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student is expected to:

(B) use slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons; and

### Additional Geometry TEKS:

### (G.7) Dimensionality and the geometry of location.

The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student is expected to:

(A) use one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures;

(C) derive and use formulas involving length, slope, and midpoint.

## Teacher Notes

### Scaffolding Questions:

#### Problem 1

##### *Axiomatic Approach*

- How would you set up a “given,” a “prove,” and a diagram that represents the conjecture?
- Which proof style would you like to use to write the proof: 2 column, flow chart, or paragraph?
- Depending on proficiency level of students, provide appropriate amounts of set up and/or proof. (See sample solution.)

##### *Coordinate Geometry Approach*

- What is the slope of  $\overline{AB}$ ?
- What must be true about the slope of the perpendicular bisector of  $\overline{AB}$ ?
- What are the coordinates of the midpoint of  $\overline{AB}$ ?
- Use the slope and midpoint information to write the equation of the perpendicular bisector of  $\overline{AB}$ .
- What must be true about the coordinates of a point if it lies on this perpendicular bisector?
- What coordinate geometry formula could you use to show that the point you found is the same distance from A as from B?

#### Problem 2

##### *Transformational Approach*

- What coordinate geometry formula could you use to verify that these distances are the same?
- Find the slope of  $\overleftrightarrow{DE}$ .
- Use this information to write the equation of  $\overleftrightarrow{DE}$  in slope-intercept form.

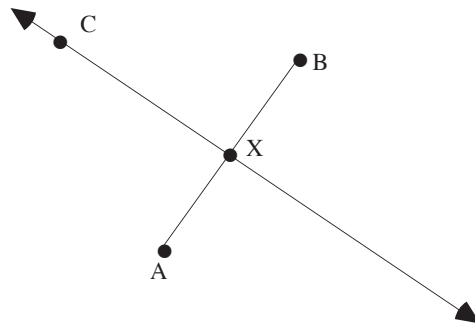
- What is the equation of the line that is perpendicular to  $\overleftrightarrow{DE}$  and passes through point A?
- What are the coordinates of point F, the intersection point of the two perpendicular lines?
- What is the distance from point A to point F (the point of intersection)?
- Pick a point on  $\overleftrightarrow{AF}$  at distance  $AF$  from point F. What are the coordinates of the point you picked?

**Sample Solutions:**

**Problem 1**

*Axiomatic Approach*

Given: Point C on  $\overleftrightarrow{CX}$ , the perpendicular bisector of  $\overline{AB}$ .  
 Prove:  $CA = CB$



Proof:

$AX = XB$	(Definition of perpendicular bisector)
$m\angle AXC = m\angle BXC = 90^\circ$	(Definition of perpendicular bisector)
$CX = CX$	(Reflexive property)
$\triangle AXC \cong \triangle BXC$	(SAS)
Therefore $CA = CB$	(Corresponding parts of congruent triangles are congruent)

*Coordinate Geometry Approach*

a) The midpoint of  $\overline{AB}$  is (5,2).

The line passing through the points A(2,-7) and B(8,11) will have slope  $\frac{11 - (-7)}{8 - 2} = \frac{18}{6} = 3$ .

(G.9) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student is expected to:

(A) formulate and test conjectures about the properties of parallel and perpendicular lines based on explorations and concrete models;

(G.10) **Congruence and the geometry of size.** The student applies the concept of congruence to justify properties of figures and solve problems.

The student is expected to:

(B) justify and apply triangle congruence relationships.

**Connections to TAKS:**

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 7: The student will demonstrate an understanding of two-and three-dimensional representations of geometric relationships and shapes.

The slope of a perpendicular line is the opposite reciprocal, or  $-\frac{1}{3}$ .  
The perpendicular bisector of  $\overline{AB}$  must have slope  $-\frac{1}{3}$  and pass through (5,2).  
Using point slope form, the equation of the perpendicular bisector is:

$$y - 2 = -\frac{1}{3}(x - 5)$$

or

$$y = -\frac{1}{3}x + \frac{11}{3}$$

b) Choose a value for  $x$ , and solve for  $y$ .

$$x = 2, y = -\frac{1}{3}(2) + \frac{11}{3} = 3$$

Point D(2, 3) is a point on the perpendicular bisector  $\overline{AB}$ .

c) Using the distance formula:

$$DA = \sqrt{(2-2)^2 + (3-(-7))^2} = 10$$

$$DB = \sqrt{(2-8)^2 + (3-11)^2} = 10$$

Therefore,  $DA = DB$ , and point D satisfies the conditions of the conjecture.

Note: Showing that one point works does not prove the statement in general. To prove in general select any point  $E(x,y)$  on the line

$$y = -\frac{1}{3}x + \frac{11}{3}$$

$$EA = \sqrt{(2-x)^2 + (-7 - (-\frac{1}{3}x + \frac{11}{3}))^2} =$$

$$\sqrt{(2-x)^2 + (-7 + \frac{1}{3}x - \frac{11}{3})^2} =$$

$$\sqrt{(2-x)^2 + (\frac{1}{3}x - \frac{32}{3})^2} =$$

$$\sqrt{4 - 4x + x^2 + \frac{1}{9}x^2 - \frac{64}{9}x + \frac{1024}{9}} =$$

$$\sqrt{\frac{10}{9}x^2 - \frac{100}{9}x + \frac{1060}{9}}$$

$$\begin{aligned} EB &= \sqrt{(8-x)^2 + \left(11 - \left(-\frac{1}{3}x + \frac{11}{3}\right)\right)^2} = \\ &= \sqrt{(8-x)^2 + \left(11 + \frac{1}{3}x - \frac{11}{3}\right)^2} = \\ &= \sqrt{(8-x)^2 + \left(\frac{1}{3}x + \frac{22}{3}\right)^2} = \\ &= \sqrt{64 - 16x + x^2 + \frac{1}{9}x^2 + \frac{44}{9}x + \frac{484}{9}} = \\ &= \sqrt{\frac{10}{9}x^2 - \frac{100}{9}x + \frac{1060}{9}} \end{aligned}$$

$EA = EB$  for any point E on  $\overline{CX}$

**Problem 2**

*Transformational Approach*

a) Using the distance formula:

$$AD = \sqrt{(2-8)^2 + (-7-1)^2} = 10$$

$$BD = \sqrt{(8-8)^2 + (11-1)^2} = 10$$

$$AE = \sqrt{(2-2)^2 + (3-(-7))^2} = 10$$

$$BE = \sqrt{(2-8)^2 + (3-11)^2} = 10$$

Therefore  $AD = BD$ , and  $AE = BE$ . This makes D and E both equidistant from A and B.

b) The slope of line  $\overleftrightarrow{DE}$  is  $\frac{3-1}{2-8} = \frac{2}{-6} = -\frac{1}{3}$ .

Using point slope form, the equation of line  $\overleftrightarrow{DE}$  is:

$$y-1 = -\frac{1}{3}(x-8)$$

or

$$y = -\frac{1}{3}x + \frac{11}{3}$$

c) The line that is perpendicular to line  $\overleftrightarrow{DE}$  and passes through A(2,-7) must have a slope that is the opposite reciprocal of  $-\frac{1}{3}$ . Using point slope formula, the equation of that line is:

$$y - (-7) = 3(x - 2). \text{ In slope intercept form: } y = 3x - 13.$$

The point of intersection of the two lines can be found by algebraically solving both equations for both variables:

$$-\frac{1}{3}x + \frac{11}{3} = 3x - 13$$

$$x = 5 \text{ and } y = 2$$

The intersection point has coordinates F(5, 2).

$$\text{The distance } AF = \sqrt{(5-2)^2 + (2-(-7))^2} = \sqrt{90} = 3\sqrt{10}.$$

We want the coordinates of the reflection image of point A. This point is  $3\sqrt{10}$

units from point F along  $\overleftrightarrow{AF}$ . To find the coordinates of the point A(x,y), use the

distance formula and the equation of  $\overleftrightarrow{AF}$ .

$$3\sqrt{10} = \sqrt{(x-5)^2 + (y-2)^2}$$

Since  $y = 3x - 13$ ,  $3\sqrt{10} = \sqrt{(x-5)^2 + (3x-13-2)^2}$

Solve this equation for  $x$ :  $3\sqrt{10} = \sqrt{x^2 - 10x + 25 + 9x^2 - 90x + 225}$

$$90 = 10x^2 - 100x + 250$$

$$0 = 10x^2 - 100x + 160$$

$$0 = 10(x^2 - 10x + 16)$$

$$0 = 10(x - 8)(x - 2)$$

$$x = 8 \text{ or } x = 2$$

When  $x = 2$ ,  $y = -7$ . These are the coordinates of point A.

When  $x = 8$ ,  $y = 11$ . These are the coordinates of point A', the reflection of point A across  $\overleftrightarrow{DE}$ .

But these are precisely the coordinates of point B. Point B is, therefore, the reflection image of point A across  $\overleftrightarrow{DE}$ . This makes  $\overleftrightarrow{DE}$  the perpendicular bisector of  $\overline{AB}$ .

**Extension Question:**

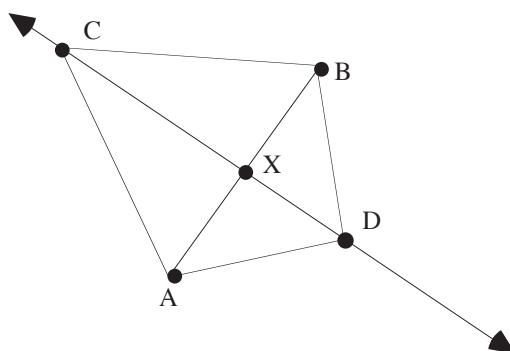
- Verify the converse of the conjecture using an Axiomatic Approach.

*Converse of Conjecture: If a point is equidistant from the two endpoints of a line segment, then the point is on the segment's perpendicular bisector.*

Given:  $CB = CA$

$$DB = DA$$

Prove:  $\overleftrightarrow{CD}$  is the perpendicular bisector of  $\overline{AB}$ .



**Proof:**

$CA = CB$ and $DA = DB$	(Given)
This makes quadrilateral $ACBD$ a kite	(Definition of a kite)
Then, $\overline{CX} \perp \overline{AB}$	(Diagonals of a kite are perpendicular)
This makes right $\triangle ACX \cong$ right $\triangle BCX$	(HL)
and $AX = XB$	(CPCTC)
Therefore, $\overline{CX}$ is the perpendicular bisector of segment $AB$ .	(Definition of perpendicular bisector)

Another approach that does not use the properties of kites, involves proving that

$\triangle CAD \cong \triangle CBD$  (SSS).

Then  $\angle ACX \cong \angle BCX$  (CPCTC).

Now,  $\triangle ACX \cong \triangle BCX$  (HA or SAS).

Then establish  $AX = XB$  and  $m\angle AXC = m\angle BXC = 90^\circ$ .

Line  $\overline{CX}$  is the perpendicular bisector of segment  $\overline{AB}$ . (Definition of perpendicular bisector)



## Conjecture as Discovery and Proof as Explanation

### Problem 1

#### *Triangle Midsegment Conjecture*

Use paper, pencil, construction and measuring tools  
or  
appropriate geometry technology to complete this problem.

1. Sketch and label  $\triangle ABC$ .
2. Find and label point D (the midpoint of side AB) and point E (the midpoint of side AC).
3. Draw midsegment  $\overline{DE}$ .
4. Take and record the following measurements in centimeters and degrees.

$$DE = \underline{\hspace{2cm}} \qquad BC = \underline{\hspace{2cm}}$$

$$m \angle ADE = \underline{\hspace{2cm}} \qquad m \angle AED = \underline{\hspace{2cm}}$$

$$m \angle DBC = \underline{\hspace{2cm}} \qquad m \angle ECB = \underline{\hspace{2cm}}$$

5. Repeat steps 1 – 4 to complete the sketch, and take measurements on at least two more triangles that are different from your original triangle.

If you are working with a group, you may compare your triangle measurements with the other group members.

If you are using geometry technology, you may drag the vertices of the original triangle to generate new triangles and sets of measurements.

6. Based on your drawings and observations, complete the following conjecture:

The midsegment of a triangle is \_\_\_\_\_ to  
one side of the triangle, and it measures \_\_\_\_\_  
of that side.

## Problem 2

### *Why Is It True?*

If you completed Problem 1, you discovered two important characteristics of a triangle's midsegment:

- a) The midsegment is parallel to a side of the triangle.
- b) The midsegment is  $\frac{1}{2}$  the length of the side of the triangle it is parallel to.

You might have already known about these properties from previous lessons, or you might have even guessed what they were without drawing or measuring. But can you explain why they are true?

In order for your explanation to be fully convincing from a mathematical standpoint, it must satisfy three requirements. First, it must be logical. Second, it must consist of facts, definitions, postulates, or theorems that have been previously proven or accepted as true. Third, it must apply to all cases.

Write an explanation of why the first property of triangle midsegments is true. Your explanation must satisfy all of the above requirements. Use your knowledge of postulates and theorems about parallel lines and angle relationships to help with your explanation.

**Problem 3***A Different Look*

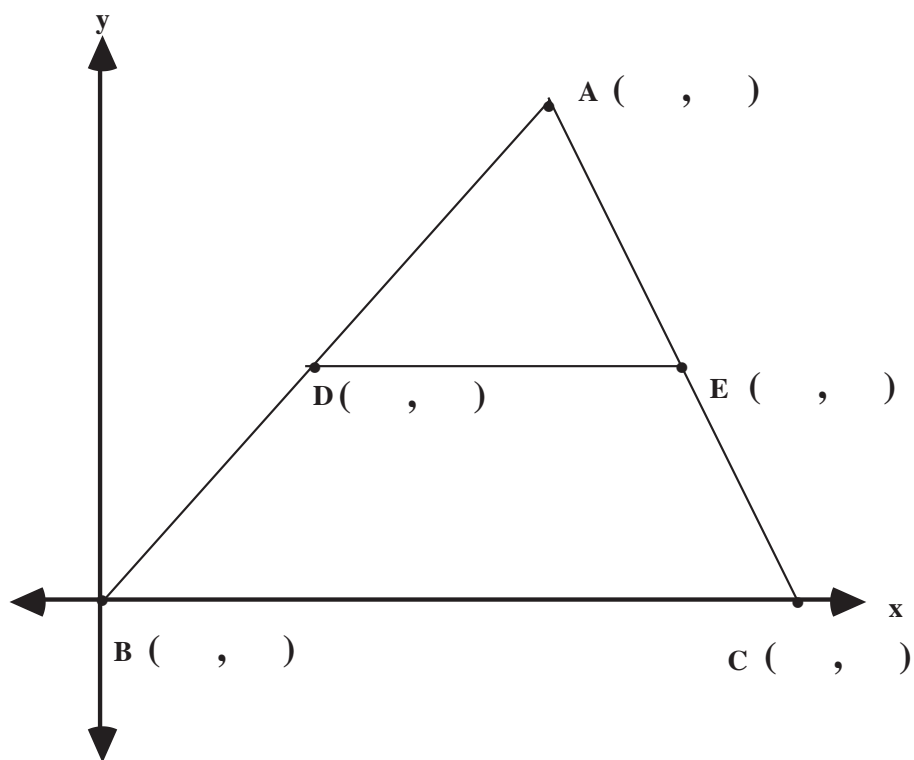
Mathematicians call explanations similar to the one you wrote in the previous problem “proofs.” In order to prove the second property about triangle midsegments, it is helpful to represent the situation in a different mathematical context.

1. Draw and label triangle  $\triangle ABC$  on graph paper. Label and record the numerical coordinates of points A, B, and C.
2. Use the midpoint formula to find and label the following: point D (the midpoint of  $\overline{AB}$ ) and point E (the midpoint of  $\overline{AC}$ ).
3. Use the distance formula to calculate the length of  $\overline{DE}$  and to calculate the length of  $\overline{BC}$ .
4. What is the relationship between the length of  $\overline{DE}$  and the length of  $\overline{BC}$  ?
5. Do the diagram and calculations above constitute a proof of the second property of triangle midsegments? Write a few sentences discussing why or why not.

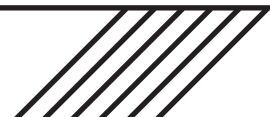
### Problem 4

#### *The General Situation*

Any triangle may be rotated and translated so that one vertex is at the origin and another vertex is on the positive  $x$ -axis.



1. Fill in the coordinates of points  $A$ ,  $B$ , and  $C$  in the above diagram using variables to represent any triangle.
2. Use the midpoint formula to calculate and fill in the coordinates of point  $D$  (the midpoint of  $\overline{AB}$ ) and point  $E$  (the midpoint of  $\overline{AC}$ ).
3. Use the distance formula to calculate the length of  $\overline{DE}$  and to calculate the length of  $\overline{BC}$ . Show all work and calculations.
4. What is the relationship between the length of  $\overline{DE}$  and the length of  $\overline{BC}$ ?
5. Do the diagram and calculations above constitute a proof of the second property of triangle midsegments? Write a few sentences discussing why or why not.





## Notes

### Materials:

Graph paper

One pencil, ruler, protractor per student, and/or

Appropriate geometry technology

### Geometry TEKS Focus:

(G.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student is expected to:

(B) make conjectures about angles, lines, polygons, circles, and three-dimensional figures and determine the validity of the conjectures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(G.3) **Geometric structure.** The student applies logical reasoning to justify and prove mathematical statements.

The student is expected to:

(B) construct and justify statements about geometric figures and their properties;

(G.7) **Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student is expected to:

(A) use one- and two-dimensional coordinate systems to represent points, lines, rays, line segments, and figures;

## Teacher Notes

This task asks the students to investigate and then prove important characteristics about a triangle's midsegment. The following performance task, Extending the Triangle Midsegment Conjecture, investigates the midsegment for other polygons. Note: Some answers to problem 1 are provided in problem 2. Teachers may want to separate problems 1 and 2 for this reason.

### Scaffolding Questions:

#### Problem 1

- Step 1:  $\triangle ABC$  can be any type of triangle. If students are working in groups, encourage different group members to draw different types of triangles.
- Step 3: What is the definition of a triangle's midsegment?
- Steps 5 and 6: What types of triangles seem to provide more convincing evidence in support of the conjecture? Why?

#### Problem 2

- What type of angles are  $\angle ADE$  and  $\angle DBC$  ?
- What type of angles are  $\angle AED$  and  $\angle ECB$  ?

#### Problem 3

- Steps 1 – 4: Students will orient their triangles on the graph paper many different ways and will soon find out that different triangle placements result in “easier” or “harder” numbers to work with.
- Is it all right to place your triangle on the graph paper in the most convenient way for the calculations to work out?
- Step 4: Can you write the relationship as a mathematical statement?
- Step 5: It can be helpful to point out that in fact the first two requirements of a mathematical proof have

been fulfilled in this case. The formulas used are accepted as true, and the diagram as well as the sequence of calculations constitute the manipulation of symbols according to logical rules. Many students do not realize this.

**Problem 4**

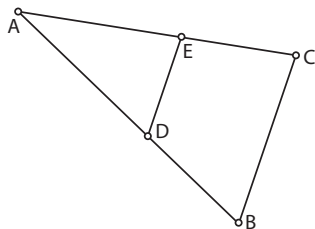
- After problem 3, students will have realized that their numerical calculations do not constitute a proof.
- What can we replace the numerical coordinates with so that the calculations will apply to any triangle?
- Why does this particular placement of the triangle make it easier to fill in the coordinates than other possible placements?
- What are the fewest number of variables that can be used to accurately label all the required coordinates?
- Step 2: What could we do to the coordinates of A, B, and C to make the calculations easier?
- What effect does multiplying a coordinate variable by a constant have on subsequent calculations? Is it mathematically all right to do this?
- Encourage students to simplify the complicated-looking expressions as much as possible. Review of notation and symbol manipulation under the radical symbol may be necessary.

**Sample Solutions:**

**Problem 1**

1 – 5. This is one possible solution.

- $AD = 3.26$  cm
- $DB = 3.26$  cm
- $AE = 2.83$  cm
- $EC = 2.83$  cm
- $m\angle ADE = 59^\circ$
- $m\angle AED = 81^\circ$
- $m\angle DBC = 59^\circ$
- $m\angle ECB = 81^\circ$



6. The midsegment of a triangle is parallel to one side of the triangle and measures one-half the length of that side.

**Additional Geometry TEKS:**

(G.3) **Geometric structure.** The student applies logical reasoning to justify and prove mathematical statements.

The student is expected to:

(C) use logical reasoning to prove statements are true and find counter examples to disprove statements that are false;

(D) use inductive reasoning to formulate a conjecture; and

(E) use deductive reasoning to prove a statement.

(G.7) **Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student is expected to:

(C) derive and use formulas involving length, slope, and midpoint.

**Connections to TAKS:**

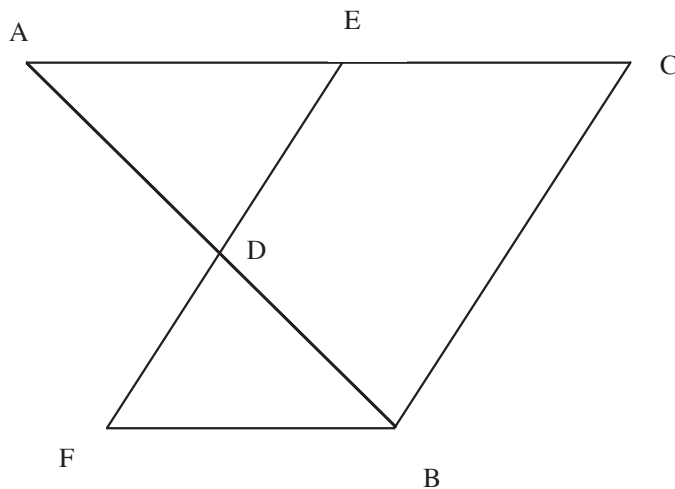
Objective 3: The student will demonstrate an understanding of linear functions.

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

**Problem 2**

A possible axiomatic proof:



Given: D is the midpoint of  $\overline{AB}$ , E is the midpoint of  $\overline{AC}$ .

Prove:  $DE = \frac{1}{2}BC$  and  $\overline{DE}$  is parallel to  $\overline{BC}$ .

Extend  $\overline{ED}$  to point F such that  $\overline{ED} \cong \overline{DF}$ .

$\overline{AE} \cong \overline{EC}$ ,  $\overline{AD} \cong \overline{DB}$  by definition of midpoint.

$\angle ADE \cong \angle BDF$  because vertical angles are congruent.

$\triangle ADE \cong \triangle BDF$  because two triangles are congruent if two sides and the included angle of one triangle are congruent to two sides and the included angle of the other triangle.

$\angle DBF \cong \angle A$  and  $\overline{AE} \cong \overline{FB}$  because corresponding parts of congruent triangles are congruent.

$\overline{AC}$  is parallel to  $\overline{BF}$  because two lines are parallel if the alternate interior angles are congruent.

$\overline{EC} \cong \overline{FB}$  by the transitive property of congruence.

BCEF is a parallelogram because the two sides of the quadrilateral are parallel and congruent.

$EF = BC$  because the opposite sides of a parallelogram are equal in length.

$FD + DE = BC$  by segment addition

$2 DE = BC$  by substitution



$$DE = \frac{1}{2} BC \text{ by division}$$

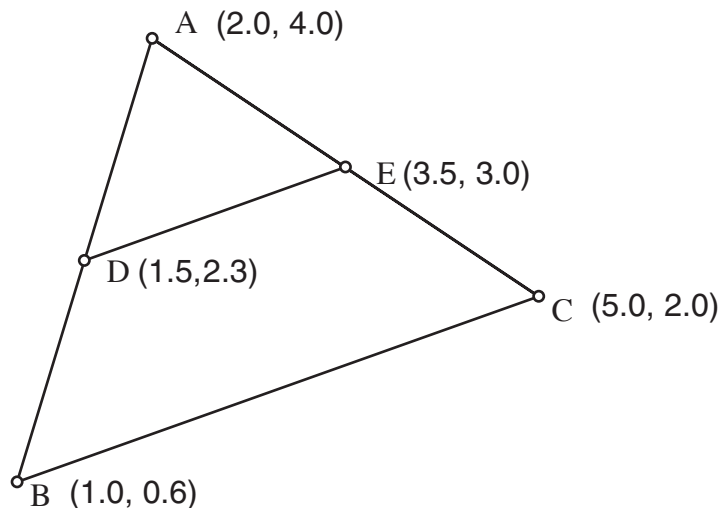
$\overline{DE}$  is parallel to  $\overline{BC}$  because opposite sides of a parallelogram are parallel.

This argument applies to the midsegment of any triangle.

**Problem 3**

One possible coordinate representation:

1 – 2.



$$3. DE = \sqrt{(3.5-1.5)^2 + (3.0-2.3)^2} = \sqrt{4+0.49} \approx 2.12$$

$$BC = \sqrt{(5.0-1.0)^2 + (2.0-0.6)^2} = \sqrt{16+3.7636} \approx 4.44$$

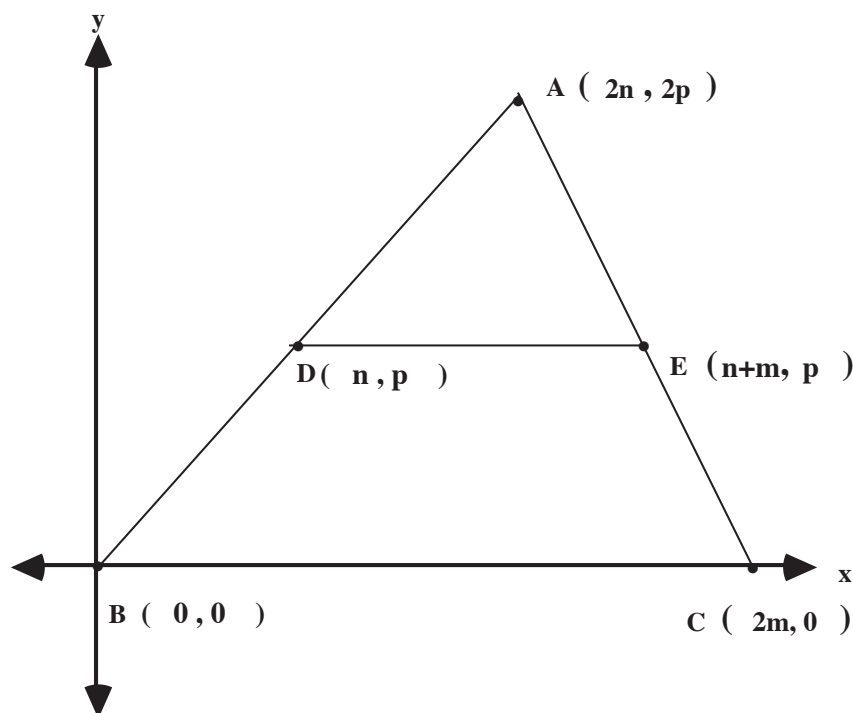
$$4. DE = \frac{1}{2} BC$$

5. The diagram and calculations establish that the midsegment is one-half the length of the side it is parallel to only for a particular triangle with specific coordinates. They do not establish that this property is true for the midsegments of all triangles.

**Problem 4**

For example:

1 – 2.



3.  $DE = \sqrt{(n+m-n)^2 + (p-p)^2} = \sqrt{m^2} = |m| = m$  because  $m > 0$ .

$BC = \sqrt{(2m-0)^2 + (0-0)^2} = \sqrt{4m^2} = |2m| = 2m$  because  $m > 0$ .

4.  $DE = \frac{1}{2} BC$

5. The diagram and the calculations do constitute a proof of the second property of triangle midsegments. The formulas used are accepted as true, and the diagram, as well as the sequence of calculations, constitutes the manipulation of symbols according to logical rules. Finally, using variables instead of specific numbers for coordinates makes the relationship true for all triangle midsegments.

**Extension Questions:**

- Write a few sentences detailing why the postulates and theorems about parallel lines and angle relationships **cannot** be used to explain why the second property of triangle midsegments described in Problem 2 (b) is true.

The postulates and theorems about parallel lines and angle relationships establish that lines are parallel given that certain angle relationships are true, or that certain angle relationships must be true given that parallel lines are cut by a transversal. The second property of triangle midsegments deals with the relationship between the lengths of the parallel segments. These postulates and theorems don't provide any information about the lengths of parallel lines.

- Using the diagram and the coordinates from problem 4, prove the first property of the triangle midsegments: the midsegment of a triangle is parallel to a side of the triangle.

The lines are parallel since it can be shown that they have the same slope.

$$\text{Slope of } \overline{DE} = \frac{p-p}{n+m-n} = 0.$$

$$\text{Slope of } \overline{BC} = \frac{0-0}{2m-0} = 0.$$

**Student Work Sample**

The work on the next two pages shows a student's approach to problems 1, 2, and 3.

This work is a good example of the criteria

- Uses geometric and other mathematical principles to justify the reasoning used to analyze the problem.

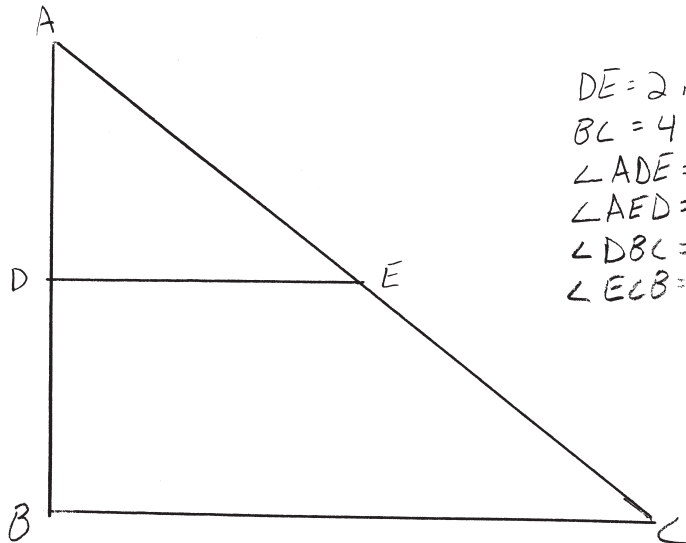
*In problems 2 and 3, this student explains his reasons for the statements he makes.*

- Makes an appropriate and accurate representation of the problem using correctly labeled diagrams.

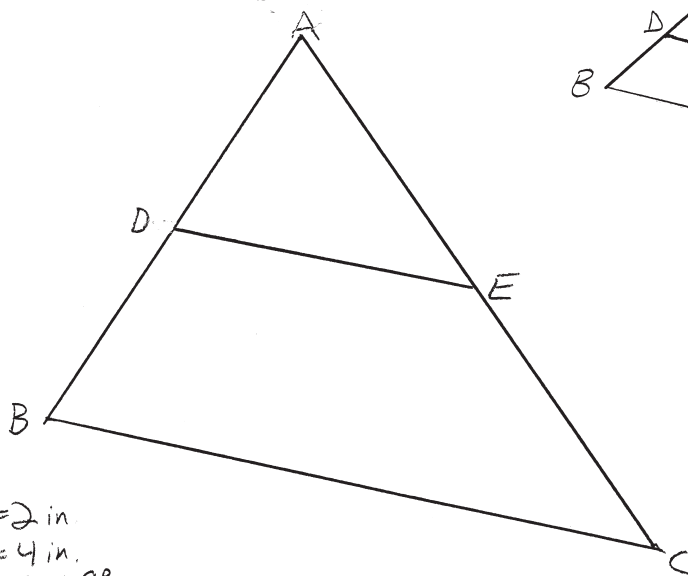
Note that the student does not explain how he got the measurements.

Conjecture as Discovery and Proof as Explanation

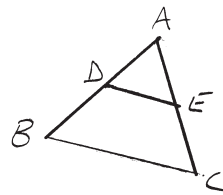
Problem 1.



$DE = 2$  in.  
 $BC = 4$  in.  
 $\angle ADE = 90^\circ$   
 $\angle AED = 32^\circ$   
 $\angle DBC = 90^\circ$   
 $\angle ECB = 32^\circ$



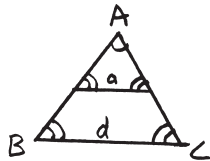
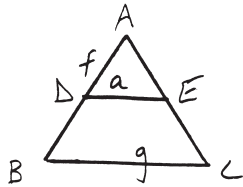
$DE = 2$  in.  
 $BC = 4$  in.  
 $\angle ADE = 69^\circ$   
 $\angle AED = 44^\circ$   
 $\angle DBC = 69^\circ$   
 $\angle ECB = 44^\circ$



$DE = \frac{1}{2}$  in.  
 $BC = 1$  in.  
 $\angle ADE = 50^\circ$   
 $\angle AED = 58^\circ$   
 $\angle DBC = 50^\circ$   
 $\angle ECB = 58^\circ$

The midsegment of a triangle is proportional to one side of the triangle and measures  $\frac{1}{2}$  of that side.

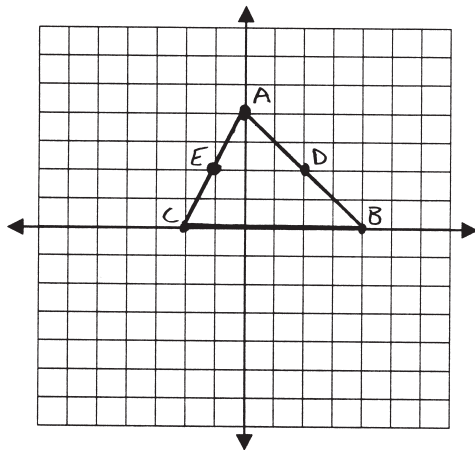
Problem 2.



Since line  $a$  is the mid-segment of triangle  $B$ , it bisects  $AB$  and  $AC$ . Therefore,  $\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{2}$ .  $\triangle ABC$  and  $\triangle ADE$  share  $\angle A$ , so by the SAS theorem,  $\triangle ABC \sim \triangle ADE$

Since they are  $\sim$ , their angles are  $\cong$ . If a line intersects 2 other lines and corresponding angles are  $\cong$ , then the lines are  $\parallel$ , so line  $a$  and line  $d$  are  $\parallel$ .

3.



$$\begin{aligned} A &= (0,4) \\ B &= (4,0) \\ C &= (-2,0) \\ D &= (2,2) \\ E &= (-1,2) \end{aligned}$$

$$\overline{DE} = \sqrt{(2 - (-1))^2 + (2 - 2)^2} = \sqrt{9} = 3$$

$$\overline{CB} = \sqrt{(4 - (-2))^2 + (0 - 0)^2} = \sqrt{36} = 6$$

$$\overline{DE} = \frac{1}{2} \overline{CB}$$

The diagram does not constitute a proof because it is only one situation. For it to be a theorem, it must work in all situations.

## Extending the Triangle Midsegment Conjecture

This performance task investigates the relationship between a polygon's diagonals and the polygon formed by connecting the midpoints of its sides. We begin the investigation by considering a quadrilateral.

### Problem 1

Use paper, pencil, and construction and measuring tools  
or  
appropriate geometry technology to complete this problem.

1. Sketch and label quadrilateral ABCD.
2. Draw all possible diagonals for quadrilateral ABCD. In this case there are two diagonals, AC and BD.
3. Create a polygon (in this case a quadrilateral) by connecting the midpoints of the sides of quadrilateral ABCD.
4. Take appropriate measurements, and write a conjecture that relates the perimeter of the midpoint quadrilateral to the diagonals of the original quadrilateral.
5. Repeat steps 1 – 4 to complete the sketch and take measurements on at least two more quadrilaterals that are different from your original quadrilateral.

If you are working with a group, you may compare your quadrilateral measurements with measurements found by the other group members.

If you are using geometry technology, you may drag the vertices of the original quadrilateral to generate new quadrilaterals and sets of measurements.

**Problem 2**

In the previous task, Conjecture as Discovery and Proof as Explanation, you discovered and proved the triangle midsegment conjecture:

The midsegment of a triangle is parallel to one side of the triangle and measures half the length of that side.

Since you have proved this conjecture, you can use it to help prove and explain why other conjectures may or may not be true.

Use the triangle midsegment conjecture to write an explanation of why your quadrilateral conjecture is true. Provide a diagram with your explanation.

**Problem 3**

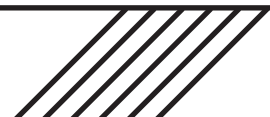
Do you think your conjecture will be true for other polygons besides quadrilaterals? Why or why not?

Sketch, measure, and investigate the relationship between a polygon's diagonals and the polygon formed by connecting the midpoints of its sides. Start with a pentagon, then a hexagon, etc.

**Problem 4**

Modify your original conjecture (or write new conjectures) to take into account other polygons besides quadrilaterals.







## Notes

### Materials:

Paper and graph paper

One pencil, ruler, and protractor per student, and/or

Appropriate geometry technology

### Geometry TEKS Focus:

(G.3) **Geometric structure.** The student applies logical reasoning to justify and prove mathematical statements.

The student is expected to:

(B) construct and justify statements about geometric figures and their properties;

(E) use deductive reasoning to prove a statement.

### Additional Geometry TEKS:

(G.3) **Geometric structure.** The student applies logical reasoning to justify and prove mathematical statements.

The student is expected to:

(C) use logical reasoning to prove statements are true and find counter examples to disprove statements that are false;

(D) use inductive reasoning to formulate a conjecture; and

## Teacher Notes

The previous task, Conjecture as Discovery and Proof as Explanation, asks the students to investigate and then prove important characteristics about a triangle's midsegment. This task investigates the midsegment for other polygons. The next task, Why Doesn't My Conjecture Always Work?, asks students to explain the results of their investigations for the different polygons.

### Scaffolding Questions:

#### Problem 1

- If students need additional structure during the investigation, tell them to restrict their measurements to the lengths of segments and to ignore angle measures.

#### Problem 2

- This explanation does not need to be a formal, two-column proof. Encourage students to examine a diagram of the triangle midsegment theorem and to see how this diagram applies to their quadrilateral diagram.

#### Problem 3

- Many students will assume that since other polygons have diagonals, the conjecture should “work the same way” no matter how many sides the polygon has. These investigations are a good way to get students to “listen to the math,” and to see how the geometry changes as the number of sides of the polygon increase.

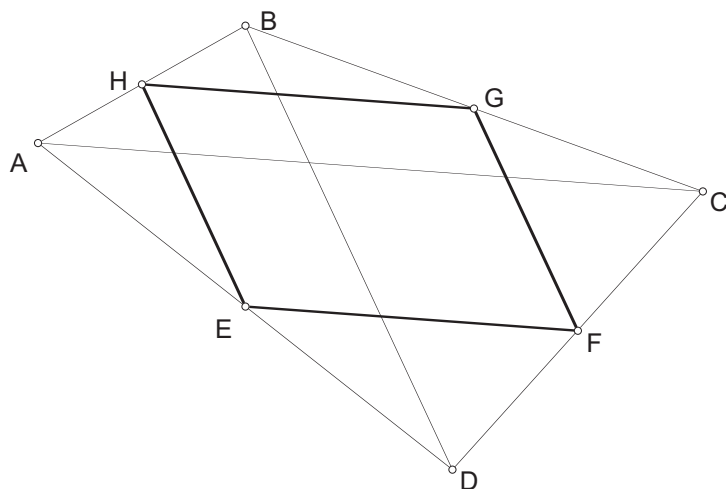
#### Problem 4

- Students should see that the initial conjecture doesn't hold for pentagons; moreover, the conjecture for pentagons doesn't hold for hexagons. Encourage students to write separate conjectures for quadrilaterals, pentagons, and polygons with more than five sides.

**Sample Solutions:**

**Problem 1**

Perimeter quadrilateral EFGH = 21.86 cm      $AC + BD = 21.86$  cm  
 $AC = 12.59$  cm  
 $BD = 9.26$  cm



Conjecture: The perimeter of the midpoint quadrilateral is equal to the sum of the lengths of the diagonals.

**Problem 2**

By the triangle midsegment conjecture:

$$GF = \frac{1}{2} BD \quad HE = \frac{1}{2} BD \quad HG = \frac{1}{2} AC$$

$$EF = \frac{1}{2} AC,$$

so

$$GF + HE + HG + EF = \frac{1}{2} BD + \frac{1}{2} BD + \frac{1}{2} AC + \frac{1}{2} AC$$

$$GF + HE + HG + EF = BD + AC.$$

Therefore, the perimeter of the midpoint quadrilateral is equal to the sum of the lengths of the diagonals.

**Connections to TAKS:**

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

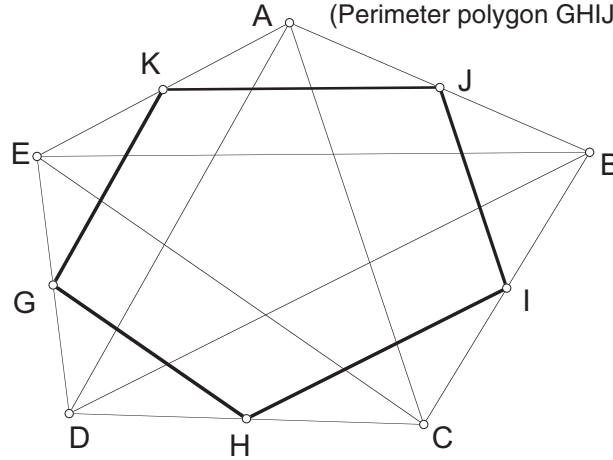
**Problem 3**

For a pentagon:

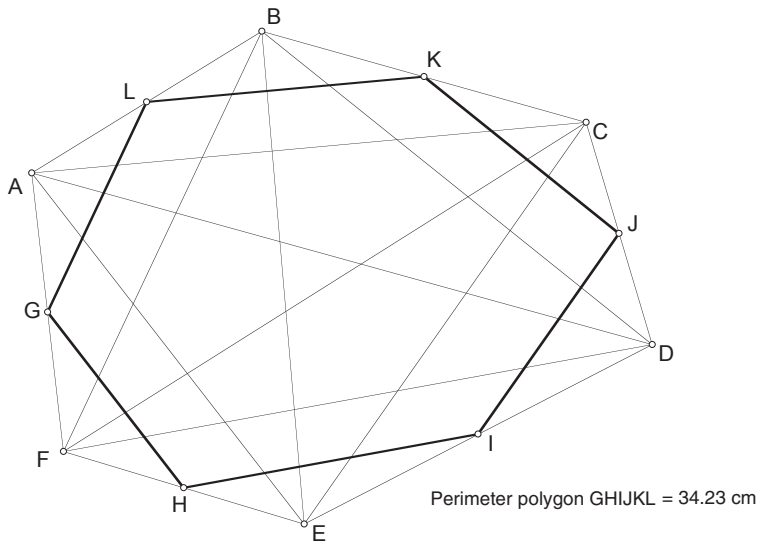
Perimeter polygon GHIJK = 20.24 cm

$EB+BD+DA+AC+CE = 40.47$  cm

$$\frac{(EB+BD+DA+AC+CE)}{(\text{Perimeter polygon GHIJK})} = 2.00$$



For a hexagon, and for polygons with more than six sides:



Perimeter polygon GHIJKL = 34.23 cm

$$AC+AD+AE+BD+BE+BF+CF+CE+DF=107.80 \text{ cm}$$

There is no relationship; the ratio between perimeter of midpoint polygon and sum of diagonals changes with different hexagons.

**Problem 4**

Conjecture: The perimeter of the midpoint quadrilateral is equal to the sum of the lengths of the diagonals.

Conjecture: The perimeter of the midpoint pentagon is one-half the sum of the lengths of the diagonals.

Conjecture: There does not appear to be any relationship between the perimeter of the midpoint polygon and the sums of the lengths of the diagonals for polygons with more than five sides.

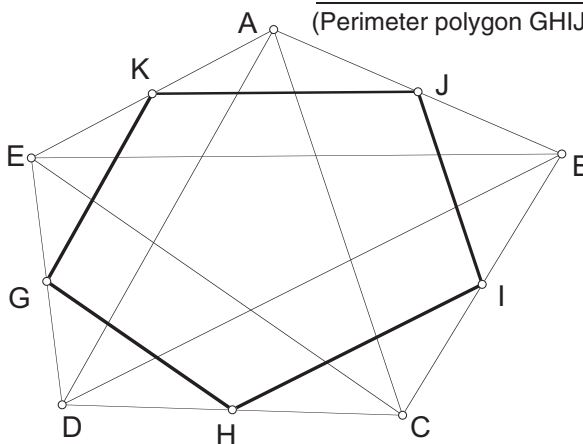
**Extension Questions:**

- Use the triangle midsegment conjecture to explain why your conjecture for pentagons must be true.

Perimeter polygon GHIJK = 20.24 cm

$EB + BD + DA + AC + CE = 40.47$  cm

$$\frac{(EB + BD + DA + AC + CE)}{(\text{Perimeter polygon GHIJK})} = 2.00$$



By the triangle midsegment conjecture:

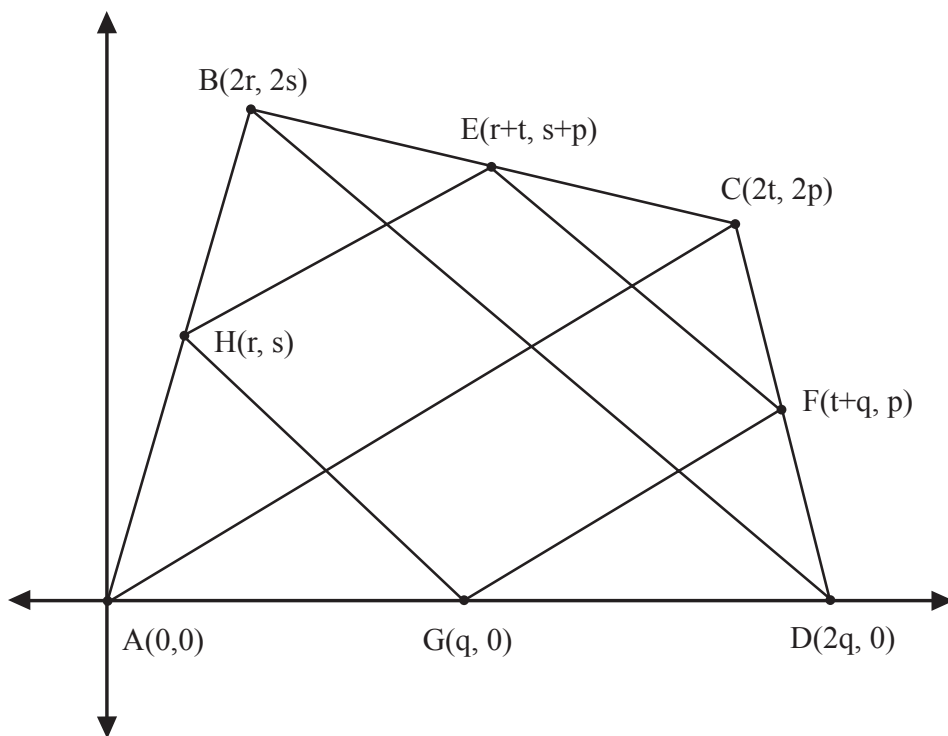
$$KJ = \frac{1}{2} EB \quad JI = \frac{1}{2} AC \quad IH = \frac{1}{2} BD \quad GH = \frac{1}{2} EC \quad GK = \frac{1}{2} AD,$$

so

$$KJ + JI + IH + GH + GK = \frac{1}{2} (EB + AC + BD + EC + AD).$$

Therefore the perimeter of the midpoint pentagon is one-half the sum of the lengths of the diagonals.

- Use coordinate geometry to prove the quadrilateral conjecture.



$$HE = \sqrt{(r+t-r)^2 + (s+p-s)^2} = \sqrt{t^2 + p^2}$$

$$FG = \sqrt{(t+q-q)^2 + p^2} = \sqrt{t^2 + p^2}$$

$$EF = \sqrt{(r+t-t-q)^2 + (s+p-p)^2} = \sqrt{(r-q)^2 + s^2}$$

$$GH = \sqrt{(r-q)^2 + s^2}$$

$$\begin{aligned} BD &= \sqrt{(2(r-q))^2 + (2s)^2} = \sqrt{4(r-q)^2 + 4s^2} = \sqrt{4((r-q)^2 + s^2)} \\ &= 2\sqrt{(r-q)^2 + s^2} \end{aligned}$$

$$AC = \sqrt{(2t)^2 + (2p)^2} = \sqrt{4t^2 + 4p^2} = \sqrt{4(t^2 + p^2)} = 2\sqrt{t^2 + p^2}$$

$$HE + GF + EF + GH = 2\sqrt{t^2 + p^2} + 2\sqrt{(r-q)^2 + s^2}$$

$$BD + AC = 2\sqrt{t^2 + p^2} + 2\sqrt{(r-q)^2 + s^2}$$

Therefore, the perimeter of the midpoint quadrilateral is equal to the sum of the lengths of the diagonals.

## Why Doesn't My Conjecture Always Work?

In the performance task, Extending the Triangle Midsegment Theorem, you investigated the relationship between a polygon's diagonals and the perimeter of the polygon formed by connecting the midpoints of its sides.

You discovered that there is no single mathematical connection between these two things that is true for all polygons; in fact, the relationship changes as the number of the polygon's sides increases.

### **For quadrilaterals:**

The perimeter of the midpoint quadrilateral is equal to the sum of the lengths of the diagonals.

### **For pentagons:**

The perimeter of the midpoint pentagon is one-half the sum of the lengths of the diagonals.

### **For polygons with more than five sides:**

There does not appear to be any relationship between the perimeter of the midpoint polygon and the sums of the lengths of the diagonals for polygons with more than five sides.

You also used the previously proven triangle midsegment conjecture to explain why the conjectures for quadrilaterals and for pentagons must be true.

### **Problem 1**

Use what you have learned in previous investigations to explain why the relationship between diagonals and midpoint polygon is different for pentagons than it is for quadrilaterals.

Provide a written explanation with diagrams.

### **Problem 2**

Use what you have learned in previous investigations to explain why there is no relationship between diagonals and midpoint polygons for shapes with more than five sides.

Provide a written explanation with a diagram(s).



## Notes

### Materials:

One pencil, ruler, and protractor per student, and/or  
Appropriate geometry technology

### Geometry TEKS Focus:

(G.3) **Geometric structure.** The student applies logical reasoning to justify and proof mathematical statements.

The student is expected to:

(C) use logical reasoning to prove statements are true and find counter examples to disprove statements that are false;

(G.5) **Geometric patterns.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to:

(B) use numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles;

### Additional Geometry TEKS:

(G.3) **Geometric structure.** The student applies logical reasoning to justify and proof mathematical statements.

The student is expected to:

(D) use inductive reasoning to formulate a conjecture; and

(E) use deductive reasoning to prove a statement.

## Teacher Notes

### Scaffolding Questions:

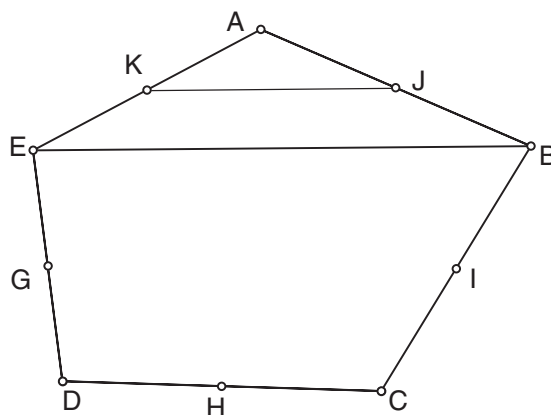
- What determines the number of times you can apply the triangle midsegment conjecture to a shape?
- What is the relationship between the segment drawn between the midpoints of consecutive sides and the diagonal drawn between two endpoints of consecutive sides?
- Complete the chart:

number of sides of polygon	number of sides of midpoint polygon	number of diagonals
4		
5		
6		

### Sample Solutions:

#### Key Concepts:

Every time a segment is drawn between the midpoints of consecutive sides of a polygon, it creates the midsegment of a triangle. This triangle's two sides are the consecutive sides of the polygon, and its third side is the diagonal drawn between the endpoints of the two consecutive sides.





For example, segment  $KJ$  is the midsegment of the triangle formed by consecutive sides  $\overline{AE}$  and  $\overline{AB}$  and diagonal  $\overline{EB}$ .

The triangle midsegment conjecture applies to the geometry of this situation, since it states that  $KJ = \frac{1}{2}EB$ .

When a midpoint polygon is formed, each of its sides is the midsegment of a triangle, and its length is therefore one-half the length of one of the original polygon's diagonals.

The reason that the relationship between the perimeter of a midpoint polygon and length of diagonals does not stay constant is that the number of times a diagonal is matched to a midsegment does not stay constant as the number of sides of the polygons increases. For some polygons there are more diagonals than sides to the midpoint polygon.

*Problem 1*

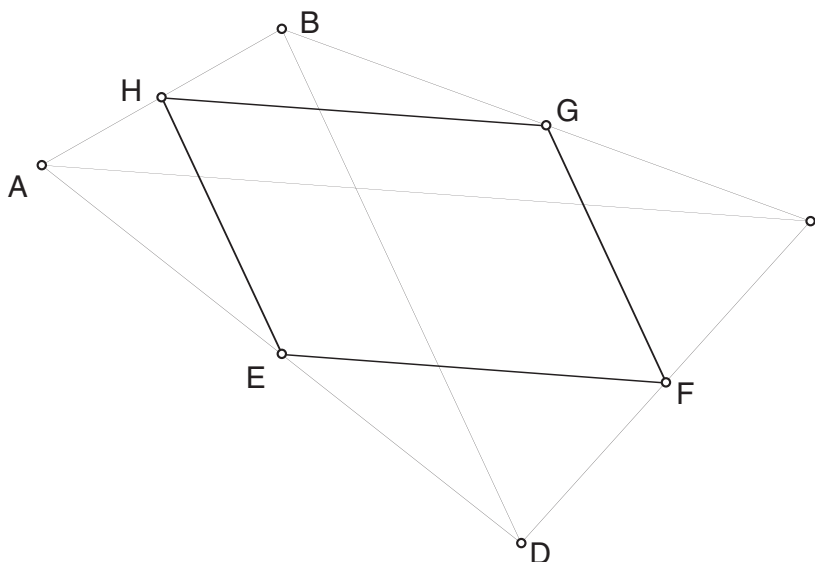
In a quadrilateral:

Perimeter quadrilateral HGFE = 21.86 cm

$BD + AC = 21.86$  cm

$BD = 9.26$  cm

$AC = 12.59$  cm



There are two diagonals and four triangle midsegments. So each diagonal gets matched with two different triangle midsegments. For example, diagonal  $\overline{AC}$  is matched with midsegment  $\overline{HG}$  and midsegment  $\overline{EF}$ .

(G.5) **Geometric patterns.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to:

(A) use numeric and geometric patterns to develop algebraic expressions representing geometric properties;

(G.9) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student is expected to:

(B) formulate and test conjectures about the properties and attributes of polygons and their component parts based on explorations and using concrete models;

**Connections to TAKS:**

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

In a quadrilateral, the triangle midsegment conjecture is applied to four different triangles corresponding to the lengths of only two different diagonals. Each diagonal, therefore, is counted twice.

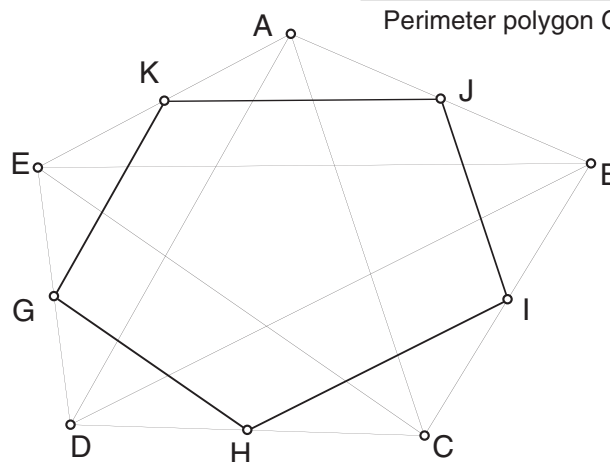
Thus, in a quadrilateral:

$$\text{perimeter of the midpoint polygon} = 2\left(\frac{1}{2}\right) (\text{sum of diagonal lengths})$$

In a pentagon, there are five diagonals and exactly five triangle midsegments. Each midsegment gets matched with one and only one diagonal.

$$\text{Perimeter polygon GHIJK} = 20.24 \text{ cm} \quad EB + BD + DA + AC + CE = 40.47$$

$$\frac{EB + BD + DA + AC + CE}{\text{Perimeter polygon GHIJK}} = 2.00$$



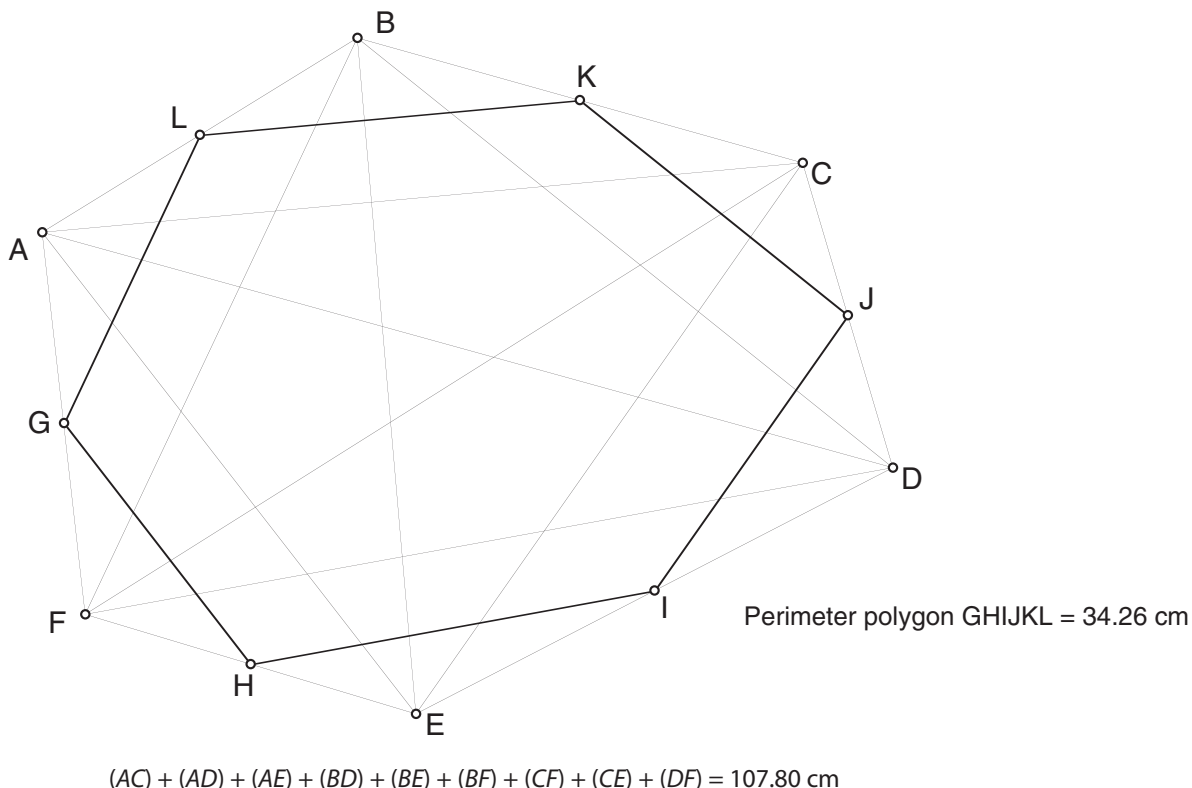
Therefore:

$$\text{perimeter of the midpoint polygon (polygon GHIJK)} = \frac{1}{2} (\text{sum of diagonal lengths})$$

*Problem 2*

For hexagons and polygons with more than five sides:

Since the number of diagonals is more than the number of sides of the midpoint polygon, some diagonals get matched with a triangle midsegment, and some don't.



In the hexagon above, diagonals  $AC$ ,  $BD$ ,  $CE$ ,  $DF$  and  $AE$  are matched to triangle midsegments. The remaining three diagonals,  $\overline{AD}$ ,  $\overline{BE}$  and  $\overline{CF}$  are not matched to a diagonal.

By the midsegment conjecture:

$$\text{perimeter of the midpoint polygon} = \frac{1}{2} (AC + BD + CE + DF + AE)$$

The conjecture, however, has nothing to say about the remaining four diagonals. Therefore, it cannot be used to establish a relationship between the perimeter of the midpoint polygon and the sum of the length of the polygon's diagonals.

The situation is the same for polygons with more sides than hexagons.

**Extension Questions:**

- Modify your conjecture about the relationship between the perimeter of a midpoint polygon and the sum of the length of the polygon's diagonals so that it applies to polygons with four or more sides.

For polygons with five or more sides:

The perimeter of the midpoint polygon is one-half the length of the sum of the lengths of the diagonals formed by connecting the endpoints of consecutive sides of the original polygon.

- Prove your conjecture for problem 1.

By the triangle midsegment conjecture, each segment connecting two midpoints is one-half the length of the diagonal connecting the endpoints of 2 consecutive sides. Since there are the same number of midpoint segments as there are diagonals:  
 perimeter of the midpoint polygon =  $\frac{1}{2}$  (sum of lengths of consecutive sides diagonals).

- Complete the chart to find a formula for the number of diagonals of a polygon in terms of  $n$ , the number of sides.

number of sides of polygon	number of sides of midpoint polygon	number of diagonals
4		
5		
6		
$n$		

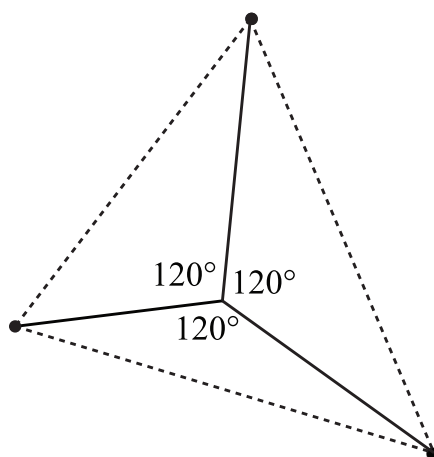
number of sides of polygon	number of sides of midpoint polygon	number of diagonals
4	4	2
5	5	5
6	6	9
$n$		

$$\text{number of diagonals} = \frac{n(n-1)}{2} - n$$



## Steiner's Point

Jakob Steiner, a nineteenth-century mathematician, found the shortest path connecting a triangle's three vertices to a central interior point. The shortest path is actually the sum of the three line segments from each vertex to this central point. Steiner proved that each segment forms a  $120^\circ$  angle with the other two segments it intersects at the central point. This central point of intersection is unique.



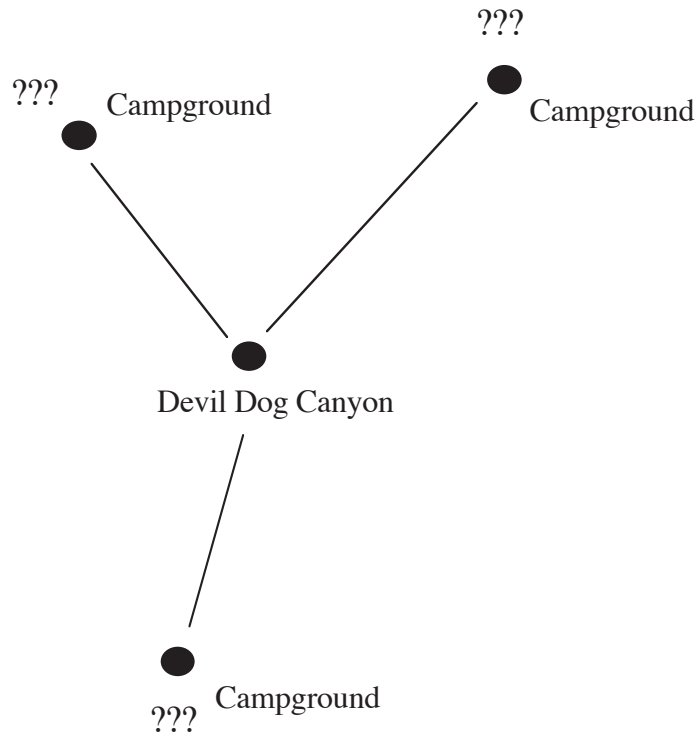
### The Situation

A developer wants to locate sites for three campgrounds so that campers have access to Devil Dog Canyon (point D), a major tourist attraction, by the shortest possible route. Billie Steiner, the developer, is going to build 18 miles of roads connecting the campgrounds to the canyon. Campground B, her high-end luxury camping resort, will have shuttles taking tourists to Devil Dog Canyon, so Steiner wants the road leading to it to be the shortest. Campground A, on the other hand, is a primitive camping area with plenty of privacy, so the road to this campground should be the longest. Campground C is a camping area that will have electric and water hookups for travel trailers and recreational vehicles, but the road leading to it does not have to be especially long or short. As a final consideration, budgeting, payroll, and scheduling will all be much simpler if the roads from Devil Dog Canyon to the campgrounds are each an integer number of miles in length.

**Problem 1**

*Ruler and Protractor*

Where should Steiner locate the campgrounds so that all the conditions are satisfied?



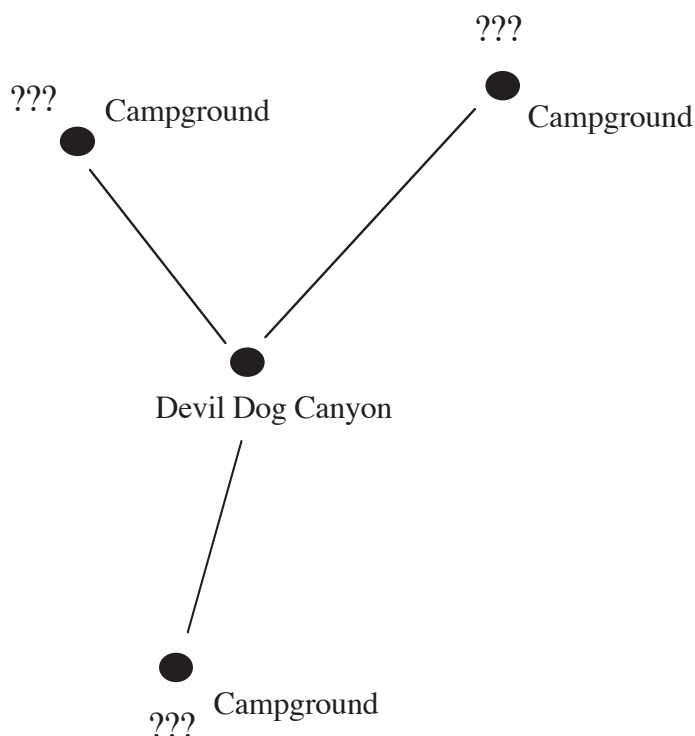
- Use a ruler and protractor to develop a possible model to represent this situation.
- Mark the position of Devil Dog Canyon (point D) and of the three campgrounds on your model.
- Provide a written explanation of how your model represents the situation, and explain how your campground sites fulfill the requirements given in the problem.



## Problem 2

### *Compass and Straightedge*

Where should Steiner locate the campgrounds so that all the conditions are satisfied?

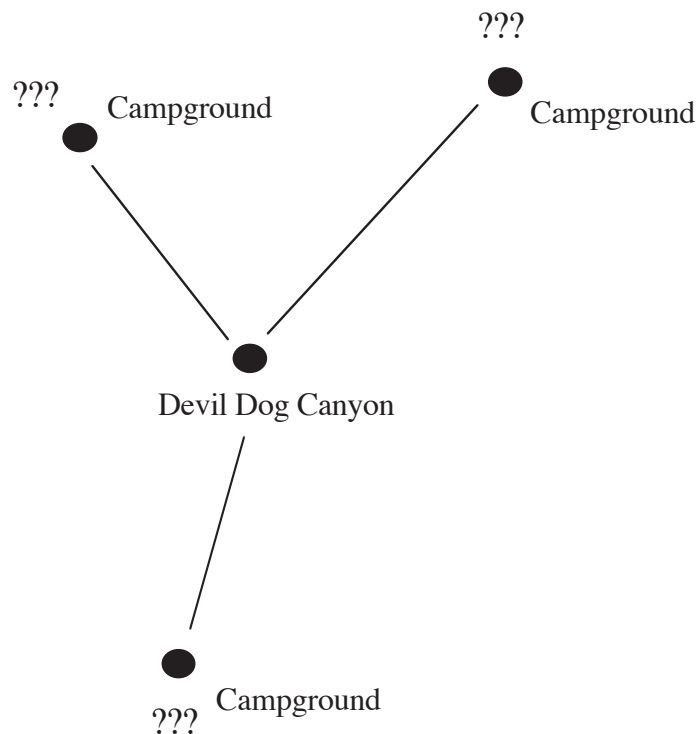


- Use a compass and straightedge to develop a possible model to represent this situation
- Mark the positions of Devil Dog Canyon and of the three campgrounds on your model.
- Provide a written explanation of how your model represents the situation, and explain how your campground sites fulfill the requirements given in the problem.

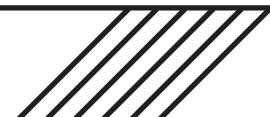
### Problem 3

#### *Graph Paper and a Coordinate System*

Where should Steiner locate the campgrounds so that all the conditions are satisfied?



- Develop a model of this situation using graph paper and an  $x,y$  coordinate system.
- Locate the positions, and label the numerical coordinates of Devil Dog Canyon and of the three campgrounds on your model. (Note: Where appropriate, express coordinates as exact values.)
- Provide a written explanation of how your model represents the situation, and explain how your campground sites fulfill the requirements given in the problem.





## Notes

### Materials:

Paper and graph paper

One ruler, protractor, compass, and straightedge per student

### Geometry TEKS Focus:

(G.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student is expected to:

(A) use constructions to explore attributes of geometric figures and to make conjectures about geometric relationships; and

(B) make conjectures about angles, lines, polygons, circles, and three-dimensional figures and determine the validity of the conjectures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

### Additional Geometry TEKS:

(G.4) **Geometric structure.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to select an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

(G.5) **Geometric patterns.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to:

(D) identify and apply patterns from right triangles to solve

## Teacher Notes

Jakob Steiner (1796-1863) was a well-known geometer. He was born in Bern, Switzerland, but spent most of his life in Germany and was a professor at the University of Berlin. He is most famous for his contributions to projective geometry. Another one of his famous theorems shows that only one given circle and a straight edge are required for Euclidean constructions. It is said that Steiner did not like algebra and analysis and felt that calculation replaces thinking while geometry stimulates thinking.<sup>1</sup>

### Scaffolding Questions:

#### Problem 1

- What are some possible integer lengths of the roadways leading from the canyon to each of the campgrounds?
- Why does  $AD + BD + DC$  represent the minimum path between the three campgrounds?

#### Problem 2

- What construction(s) could you use to make three  $120^\circ$  angles at point D?
- What construction(s) could you use to mark off distances from the canyon (point D) to the three campgrounds so that  $BD$  is the shortest and  $DA$  is the longest?
- In marking off these distances to meet the problem's requirements, which segment should you use as your unit length? Why?
- Why does  $AD + BD + DC$  represent the minimum path between the three campgrounds?

#### Problem 3

- Which point in the model would be the best choice to locate at the origin? Why?

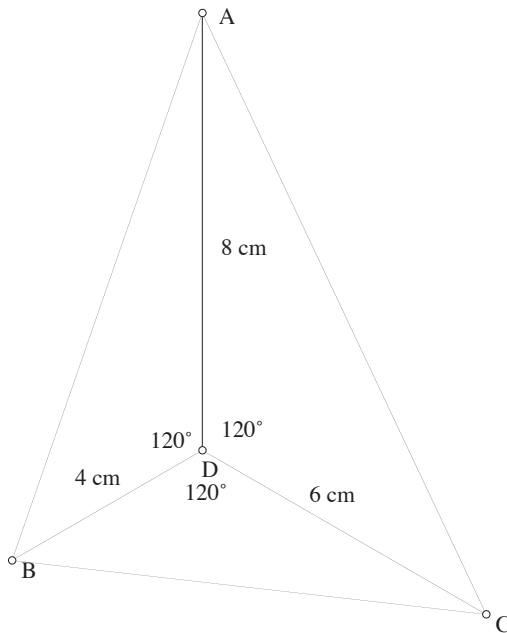
1. Jakob Steiner. The Mac Tutor History of Mathematics Archive. University of St. Andrews, Scotland. (December 1996). Retrieved July 5, 2006, from <http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Steiner.html>.

- Is it possible to orient the vertices of your triangle so that the distance from one of the campgrounds to the canyon lies along one of the axes?
- (If above is accomplished) What are the angle measures between the  $x$  axis and the segments from the canyon to each of the two other canyons?
- How can you use the properties of 30-60-90 right triangles to determine the coordinates of points B and C?
- For campgrounds B and C, which coordinate,  $x$  or  $y$ , corresponds to the short leg of the 30-60-90 right triangle? Which coordinate,  $x$  or  $y$ , corresponds to the long leg of the 30-60-90 right triangle?
- Why does  $AD + BD + DC$  represent the minimum path between the three campgrounds?

**Sample Solutions:**

*Problem 1*

There are many possible solutions. The distances from D to the triangle vertices can be any three integers that sum to 18, such that  $DB < DC < DA$ .



(Note: Distances from D to triangle vertices can be any three integers that sum to 18, such that  $DB < DC < DA$ .)

meaningful problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

**(G.7) Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student is expected to:

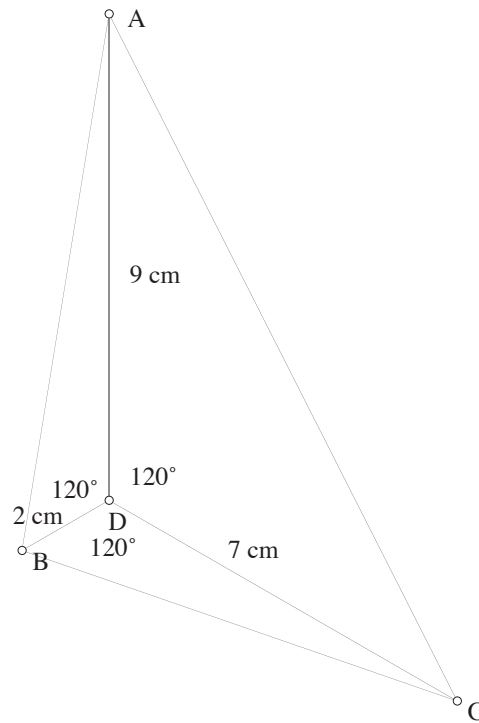
(A) use one- and two-dimensional coordinate systems to represent points, lines, rays, line segments, and figures;

**Connections to TAKS:**

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Another possible solution:

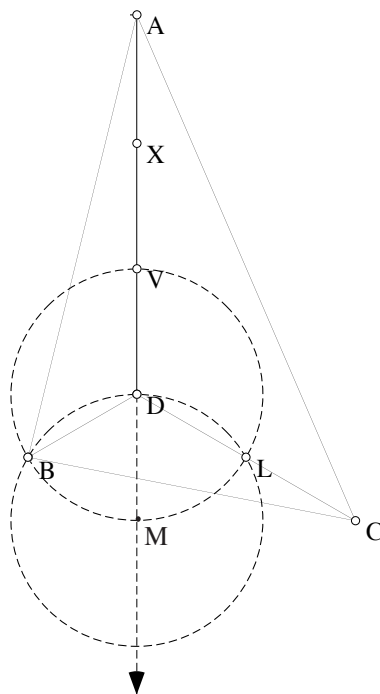


c) Here is one possible explanation for this last example:

$\overline{DA}$  was drawn with length 9 centimeters. At point D, the two 120-degree angles were measured. The third angle will also be 120 degrees.  $\overline{DB}$  was measured 2 centimeters, and  $\overline{DC}$  was measured 7 centimeters. The sum of the segments is  $9 + 2 + 7$ , or 18 centimeters. Point D meets the requirements for Steiner's minimum path because each path forms a 120 degree angle at D.

Problem 2

One possible solution:



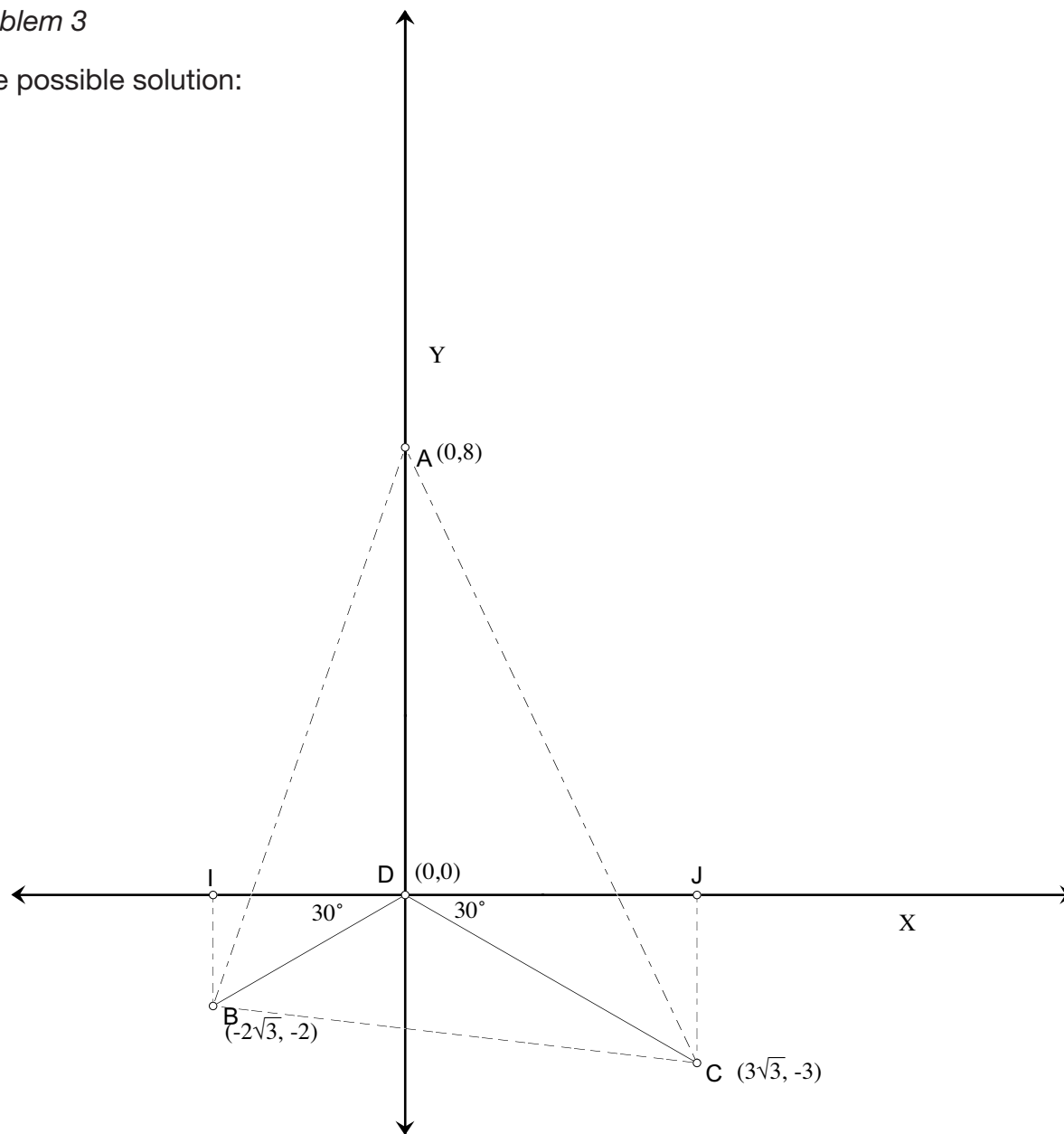
(Note: By construction  $DC = 2BD$ , and  $DA = 3BD$ . In the model  $BD = 3$  units,  $DC = 6$  units, and  $DA = 9$  units.)

- c) Begin with a point, D. Draw a line,  $\overleftrightarrow{DX}$ . Choose a radius to represent 3 miles. Use that radius to construct a circle with center D. The circle will intersect  $\overleftrightarrow{DX}$  at point V and M. Use M as a center to construct another circle with the same radius. The two circles intersect at points B and L. Draw  $\overline{BD}$  and  $\overline{DL}$ . Extend  $\overline{DL}$  to point C, such that  $DC = 2BD$ . Extend  $\overline{DV}$  to point A so that  $DA = 3BD$ .

Triangle DMB would be an equilateral triangle with three sides congruent to  $\overline{BD}$ . Therefore, angles BDM and LDM measure 60 degrees. Angle BDL measures 120 degrees. Angle BDX measures  $(180 - 60)$  degrees or 120 degrees. The point D meets the conditions of Steiner's minimum path.

Problem 3

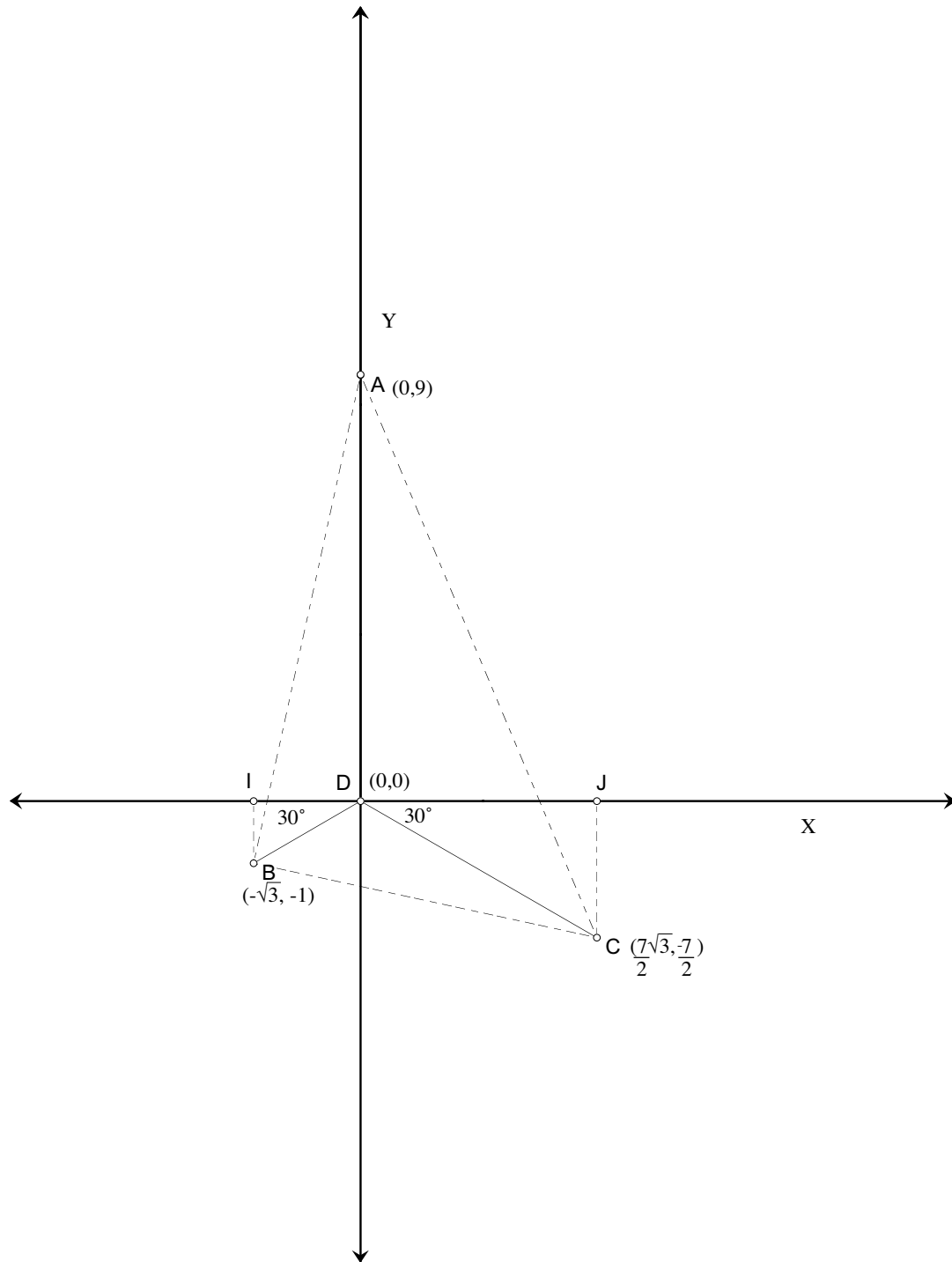
One possible solution:



Let point D represent Devil Dog Canyon and be plotted at  $(0,0)$ . Let the length of segment  $AD = 8$  and represent this on the graph by plotting point A at  $(0,8)$ . Let the length of segment  $BD = 4$ . To find the coordinates of point B create a right triangle BDI by drawing a line from B perpendicular to the x-axis. Given the fact that the angle  $ADB = 120^\circ$  and angle  $ADI = 90^\circ$ , then angle  $BDI = 30^\circ$ . Using the patterns in 30-60-90 right triangles, the coordinates of B are  $(-2\sqrt{3}, -2)$ .  $AD + BD = 12$ , therefore  $DC = 6$ . Create a right triangle CDJ by drawing a line from C perpendicular to the x-axis. Again using a 30-60-90 right triangle, the coordinates of C are  $(3\sqrt{3}, -3)$ . The sum of the distances of  $BD + CD + AD$  is  $4 + 6 + 8$  or 18 units. The points A, B, and C meet the requirements of Steiner's minimum path.



Another possible solution:

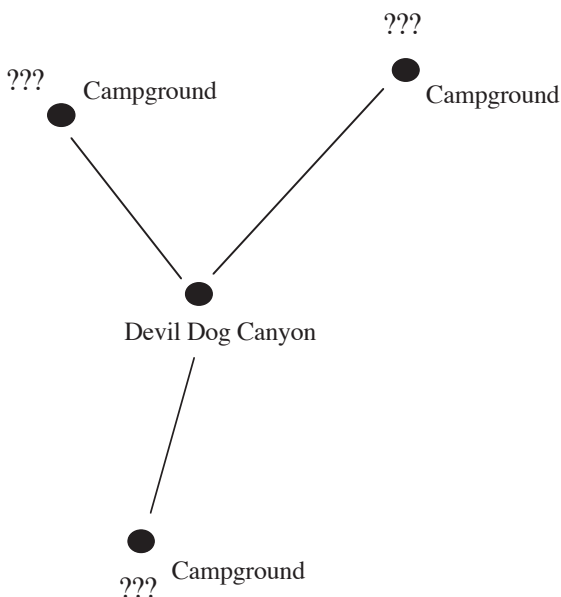




## Steiner's Point Revisited

In your work on the Steiner's Point task, you developed geometric models for locating campground sites. In that task, you used a ruler and protractor, compass and straightedge, and coordinate geometry to locate the campgrounds.

Where should Steiner locate the campgrounds so that all the conditions are satisfied?

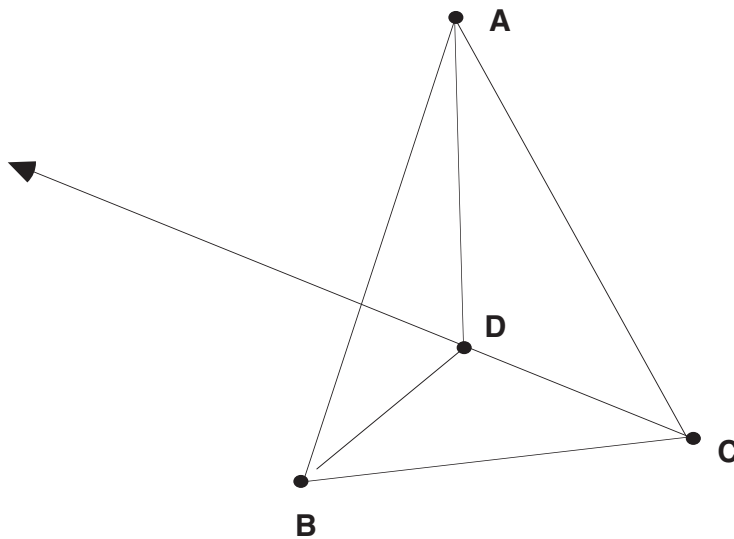


Jakob Steiner, a nineteenth-century mathematician, noticed that any three non-linear points can be used to form a triangle. He found the shortest path connecting a triangle's three vertices to a central interior point. The shortest path is actually the sum of the three line segments from each vertex to this central point. Steiner proved that each segment forms a  $120^\circ$  angle with the other two segments it intersects at the central point. This central point of intersection is unique.

## The Setup

Use your ruler and protractor model and your coordinate geometry model from the Steiner's Point task.

Draw a ray from one of the vertices of the triangle intersecting point D (that represents Devil Dog Canyon) in the triangle's interior, and then continuing through the triangle's exterior.



## Problem 1

### *Ruler and Protractor Model*

- a) Use transformations to locate and label point F on the ray so that it satisfies the following requirement:

The distance from F to the ray's endpoint = distance from Campground A to canyon + distance from Campground B to canyon + distance from Campground C to canyon.

(Note:  $FC = 18$  units. How can you use transformations to get the images of  $\overline{BD}$  and  $\overline{AD}$  to lie adjacent to each other on the ray?)

- b) Provide a written explanation of how you used transformations to locate point F.

**Problem 2***Explain Your Thinking*

Write a paragraph explaining how the path between the campgrounds and the canyon must be the minimum path between these points. Use your findings from the previous problem as support for your argument.

**Problem 3***Coordinate Geometry Model*

- a) Use your knowledge of special right triangles to locate and label the exact numerical coordinates of point F on the ray so that it satisfies the following requirement:

The distance from F to the ray's endpoint = distance from Campground A to canyon + distance from Campground B to canyon + distance from Campground C to canyon.

- b) Provide a written explanation of how you used special right triangles to find the coordinates of point F.



## Notes

### Materials:

Graph paper

One pencil, ruler, protractor, and compass per student

### Geometry TEKS Focus:

(G.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student is expected to:

(A) use constructions to explore attributes of geometric figures and to make conjectures about geometric relationships; and

(G.5) **Geometric patterns.**

The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to:

(C) use properties of transformations and their compositions to make connections between mathematics and the real world, such as tessellations; and

### Additional Geometry TEKS:

(G.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student is expected to:

(B) make conjectures about angles, lines, polygons, circles, and three-dimensional figures and determine the validity if the conjectures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(G.4) **Geometric structure.**

The student uses a variety of representations to describe geometric relationships and solve problems.

## Teacher Notes

### Scaffolding Questions:

#### Problem 1

- If we know that  $FC = 18$  cm, how is it possible to use transformations to get images of segments  $\overline{BD}$  and  $\overline{AD}$  to lie adjacent to each other on segment  $\overline{FC}$ ?
- Why would it be advantageous to do this?
- What shape contains both segment  $\overline{BD}$  and segment  $\overline{AD}$ ?
- How many degrees, and around what point, would  $\triangle ADB$  have to rotate in order for the image of segment  $\overline{AD}$  to lie on ray  $\overline{CD}$ ?
- Why is  $\triangle BDD'$  an equilateral triangle?

#### Problem 2

- What is the shortest path between points F and C?
- Express  $\overline{FC}$  as the sum of the lengths of the three segments that compose it.
- Which two of the three segments composing  $\overline{FC}$  are transformed images of segments representing roads from the canyon to the campgrounds?
- Express  $\overline{FC}$  as the sum of the lengths of the three segments leading from the canyon to the campgrounds.

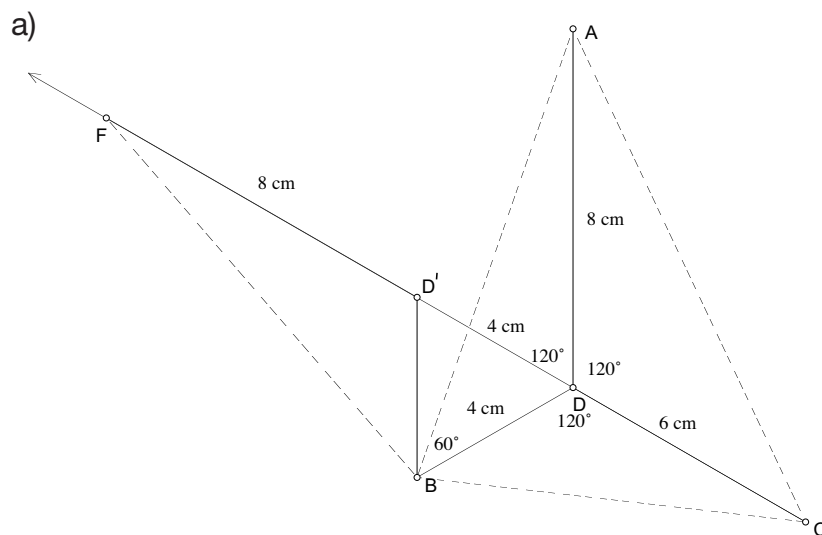
#### Problem 3

- What must the length of segment  $\overline{FD}$  be?
- Explain how to determine the measure of angle FDM.
- For point F, what coordinate,  $x$  or  $y$ , corresponds to the short leg of 30-60-90 right  $\triangle FDM$ ? What coordinate,  $x$  or  $y$ , corresponds to the long leg of 30-60-90 right  $\triangle FDM$ ?

**Sample Solutions:**

*Problem 1*

This is one possible solution.

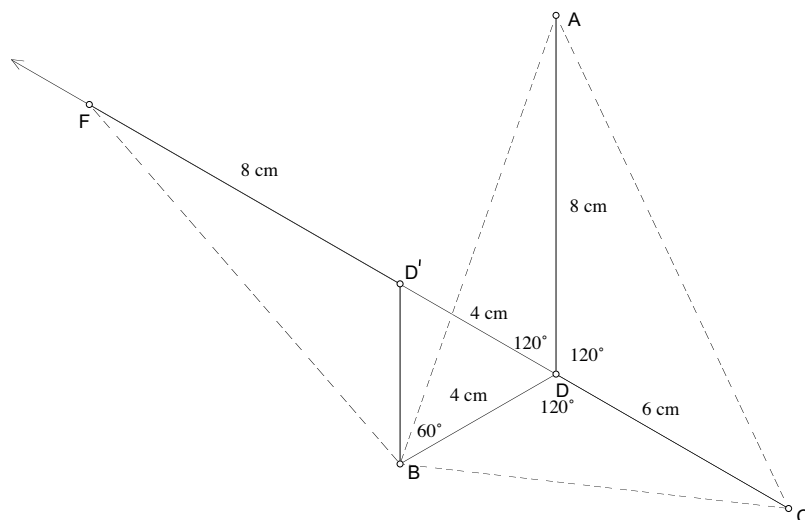


- b) Rotate  $\triangle ADB$   $60^\circ$  around point B.  $FD' = AD = 8$  cm  
 $BD = BD' = 4$  cm This makes  $\triangle BDD'$  equilateral.  
 Therefore  $DD' = 4$  cm.

$$FC = AD + DB + DC = 18 \text{ cm.}$$

*Problem 2*

For example, using the ruler and protractor model, the minimum distance path between points F and C is the straight line segment FC.



The student is expected to select an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

**(G.5) Geometric patterns.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to:  
 (D) identify and apply patterns from right triangles to solve meaningful problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

**(G.7) Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student is expected to:  
 (A) use one- and two-dimensional coordinate systems to represent points, lines, rays, line segments, and figures;

**Connections to TAKS:**  
 Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

$$FC = FD' + D'D + DC.$$

But by preservation of length under transformations,

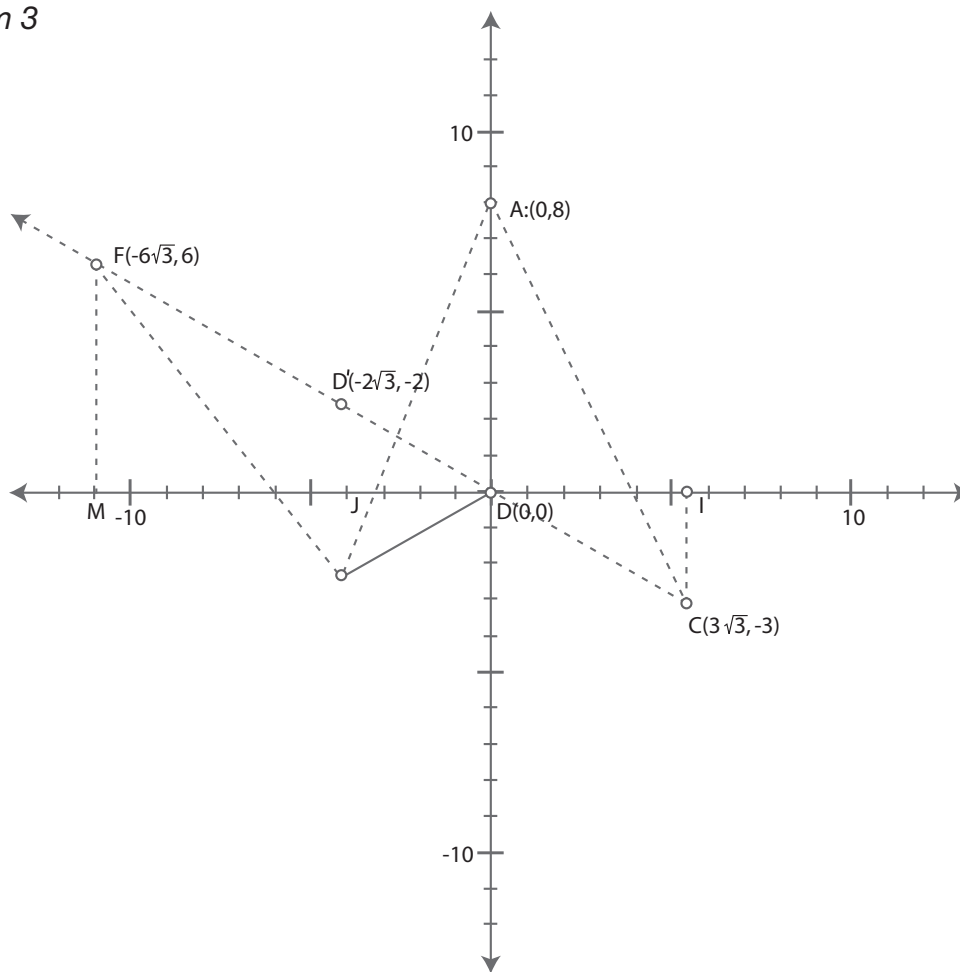
$FD' = AD$  and  $D'D = BD$ , so by substitution,

$$FC = AD + BD + DC.$$

Since  $FC$  is the shortest distance between  $F$  and  $C$  it, therefore, must be the minimum path distance between the three points  $A$ ,  $B$ , and  $C$ .

**Problem 3**

a)



b) Rotate triangle  $ADB$   $60^\circ$  around point  $B$ .  $FD' = AD = 8$  and  $BD = BD' = 4$ . This makes triangle  $BDD'$  equilateral. Therefore,  $DD' = 4$  and  $FD = 12$ . Angle  $ADD' = 60^\circ$  since triangle  $ABD$  was rotated  $60^\circ$ . Create a point  $M$  on the  $x$ -axis such that  $FM$  is perpendicular to  $MD$ . Angle  $MDF = 30^\circ$  and Angle  $FMD = 90^\circ$  therefore Triangle  $MDF$  is a  $30$ - $60$ - $90^\circ$  right triangle with  $FD = 12$ ,  $FM = 6$ , and  $MD = 6$ . Therefore the coordinates of  $F$  are  $(-6, 6)$ . Students may use the distance formula to confirm  $FC$  is equal to  $18$ . Length of  $FC = \sqrt{(3\sqrt{3} + 6\sqrt{3} + (-3 - 6))^2}$

$$\begin{aligned} & \sqrt{243 + 81} \\ & \sqrt{384} \\ & = 18 \end{aligned}$$



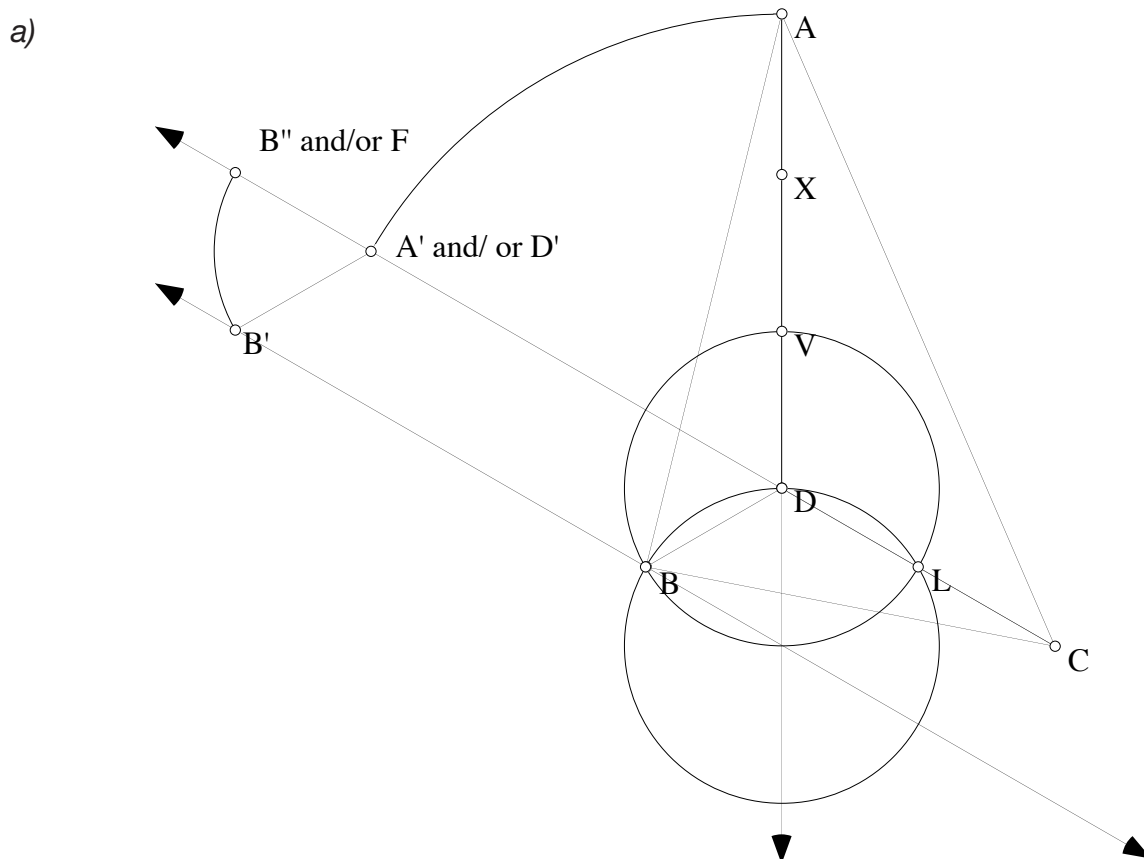
**Extension Questions**

- Use the compass and straightedge model.
- a) Use transformations to locate and label point F on the ray so that it satisfies the following requirement:

The distance from F to the ray's endpoint = distance from Campground A to canyon + distance from Campground B to canyon + distance from Campground C to canyon.

Note: For this model you are limited to your construction tools as you perform and describe the transformations.

- b) Provide a written explanation of how you used transformations to locate point F.



- b) Rotate point A about point D until the rotated point intersects the ray. Label the intersection  $A'$ .  $DA = DA'$ .

Construct a line parallel to  $\overrightarrow{DC}$  through point B. Translate point D distance  $DA'$  along  $\overrightarrow{DA'}$ . Translate point B distance  $DA'$  the parallel line through point B.  $B'D' = BD$ .

Rotate point  $B'$  about point  $A'$  until the rotated point intersects the ray.

Mark the intersection point F.

$$FA' = B'D' = BD.$$

$$FC = AD + BD + DC.$$

- In the solution to the first Extension Question, how is it possible to get the transformed image of segment BD to lie adjacent to segment  $DA'$  without constructing a line parallel to  $\overrightarrow{CD}$  through point B?

Rotate point B about point D until  $B'$ , the image point, intersects  $\overrightarrow{CD}$ . Then translate  $B'$  distance  $DA'$  along  $\overrightarrow{CD}$ .

## Walking the Archimedean Walk

Most geometry students know where the value of  $\pi$  comes from—their calculators. Most geometry students probably also realize that the number their calculator gives them is really an approximation of the value of  $\pi$ —the constant ratio between a circle’s circumference and its diameter:

$$\pi = \frac{C}{d}$$

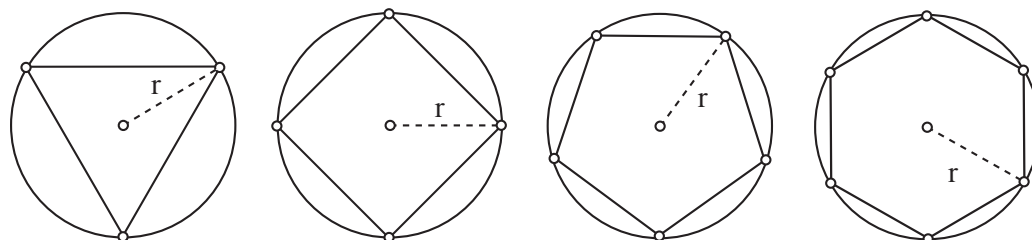
During the course of human history, diverse cultures throughout the world were aware of this constant ratio. The attempt to fix its exact value has been a vexing problem that has occupied many mathematical minds over the centuries. (See, for example, *A History of Pi* by Petr Beckmann, St. Martin’s Griffin, 1976.)

In our Western cultural tradition, the historical record tells us that Archimedes was the first person to provide a mathematically rigorous method for determining the value of  $\pi$ .

In this assessment, you will have to retrace his footsteps in order to demonstrate a solid understanding of where that number comes from when you push the “ $\pi$ ” button on your calculator.

A logical starting place for determining  $\pi$  is to measure the circumferences and diameters of many circles and then calculate the ratio  $C/d$  based upon those measurements. You may have done a measurement activity like this in your geometry class. Archimedes realized that this method would always be limited by the precision of the people doing the measuring and by the accuracy of the measuring devices they were using. He sought a way to fix the value of  $\pi$  that was based upon direct calculation rather than upon measurement.

Archimedes's approach involved inscribing regular polygons in circles. He then considered what " $\pi$ " would be for each of the inscribed regular polygons. You will model his approach by examining the figures below in the problems for this assessment.



### Problem 1

1. Use construction tools, circular geoboards, circular dot paper, or appropriate geometry technology to inscribe an equilateral triangle in a circle with a radius of 2 units.
2. Make appropriate calculations (no measuring allowed!) in order to determine the value of  $\pi$  based upon an inscribed equilateral triangle.

### Problem 2

1. Use construction tools, circular geoboards, circular dot paper, or appropriate geometry technology to inscribe a square in a circle with a radius of 2 units.
2. Make appropriate calculations (no measuring allowed!) in order to determine the value of  $\pi$  based upon an inscribed square.

### Problem 3

1. Use construction tools, circular geoboards, circular dot paper, or appropriate geometry technology to inscribe a regular hexagon in a circle with a radius of 2 units.
2. Make appropriate calculations (no measuring allowed!) in order to determine the value of  $\pi$  based upon an inscribed regular hexagon.

**Problem 4**

1. Complete the table below, and summarize your findings.

Number of sides of the inscribed polygon	Measure of the central angle	Perimeter of inscribed polygon	Diameter of inscribed polygon	Approximation for $\pi$
3			4	
4			4	
6			4	

2. Does this method overestimate or underestimate the value of  $\pi$ ? Will this method result in an exact value for  $\pi$ ?
3. Write a few sentences and provide a diagram to answer question 2.

**Problem 5**

Write a few sentences explaining the basics of the method Archimedes used.



## Notes

### Materials:

Construction tools, circular geoboards, circular dot paper, or geometry software

### Geometry TEKS Focus:

(G.1) **Geometric structure.** The student understands the structure of, and relationships within, an axiomatic system.

The student is expected to:

(B) recognize the historical development of geometric systems and know mathematics is developed for a variety of purposes; and

(G.5) **Geometric patterns.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to:

(B) uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.

### Additional Geometry TEKS:

(G.5) **Geometric patterns.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to:

(A) uses numeric and geometric patterns to develop algebraic expressions representing geometric properties;

(D) identify and apply patterns from right triangles to solve meaningful problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

## Teacher Notes

The next performance task, Talk the Archimedean Talk, is an extension of this task. It requires the student to repeat this activity for a dodecagon.

### Scaffolding Questions:

#### *Problems 1, 2, and 3*

- What special right triangle is formed by the radius and the apothem of the inscribed polygon?

#### *Problem 4*

- Will the perimeter of the inscribed polygon be greater than, less than, or equal to the circumference of the circle?
- Will the ratio  $\frac{\text{perimeter}}{\text{diameter}}$  be greater than, less than, or equal to the ratio  $\frac{\text{circumference}}{\text{diameter}}$  ?
- Will the perimeter of the inscribed polygon ever be equal to the circumference of the circle?

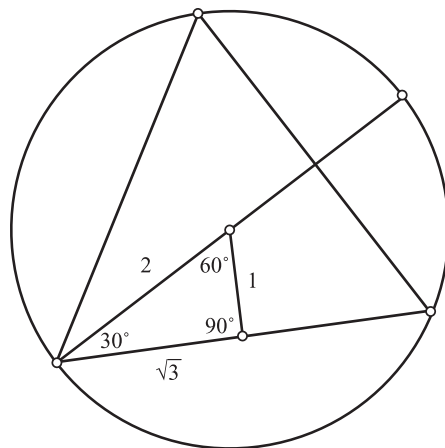
#### *Problem 5*

- If you consider the inscribed regular polygon to be a “primitive circle,” what measure in the polygon corresponds to the circumference of the circle?
- If you consider the inscribed regular polygon to be a “primitive circle,” what measure in the polygon corresponds to the diameter of the circle?
- What will happen to the ratio  $\frac{\text{perimeter}}{\text{diameter}}$  as the number of sides of the polygon increases?

**Sample Solutions:**

*Problem 1*

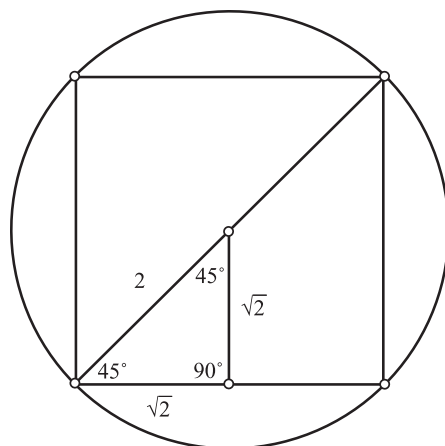
1.



2.  $\frac{\text{perimeter}}{\text{diameter}} = \frac{6\sqrt{3}}{4} = \frac{3\sqrt{3}}{2} \approx 2.5981$

*Problem 2*

1.



2.  $\frac{\text{perimeter}}{\text{diameter}} = \frac{8\sqrt{2}}{4} = 2\sqrt{2} \approx 2.8284$

**Connections to TAKS:**

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

**One teacher says . . .**

“I introduced it to my students as a Gold Medal problem (which they knew required a written, detailed response.) They worked on it independently for 30 minutes, and then they had an opportunity to work in a group setting.

Many students extended the problem naturally to see how close they could come to  $\pi$ .

I used geoboard circular dot paper and that worked well. The students saw the natural progression to a dodecagon.”

**One student says . . .**

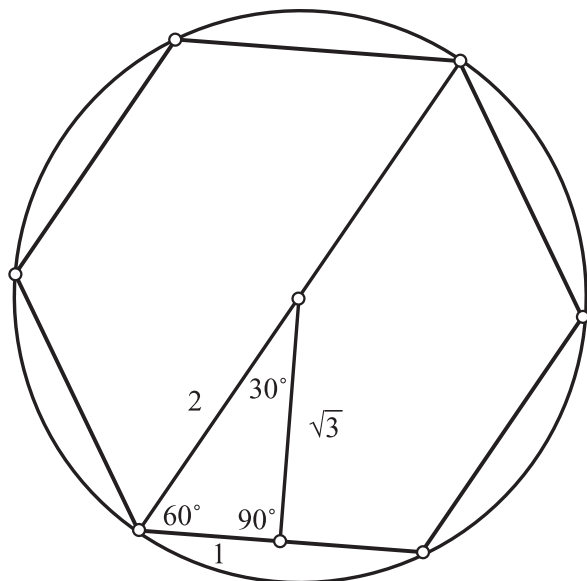
“This problem looked very complex before I started, but when I began working, it helped create itself and turned out to be relatively simple with the proper procedures. I learned to think differently and used sub-problems to meet my conclusions. I don’t believe that I’ve ever done a similar problem, but I know that if I do, I will know how to start. The sub-problems really made it easy for me to follow where the problem led.”

**One student says . . .**

“Archimedes must have been a really smart man and must have had a lot of time on his hands to figure out this problem, especially without calculators.”

Problem 3

1.



2.  $\frac{\text{perimeter}}{\text{diameter}} = \frac{12}{4} = 3$

Problem 4

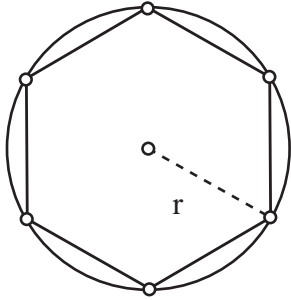
1.

Number of sides of the inscribed polygon	Measure of the central angle	Perimeter of inscribed polygon	Diameter of inscribed polygon	Approximation for $\pi$
3	$120^\circ$	$6\sqrt{3}$	4	2.5981
4	$90^\circ$	$8\sqrt{2}$	4	2.8284
6	$60^\circ$	12	4	3.000

2. This method underestimates the value of  $\pi$ . The perimeter of the inscribed regular polygon is always less than the circumference of the circle it is inscribed within.



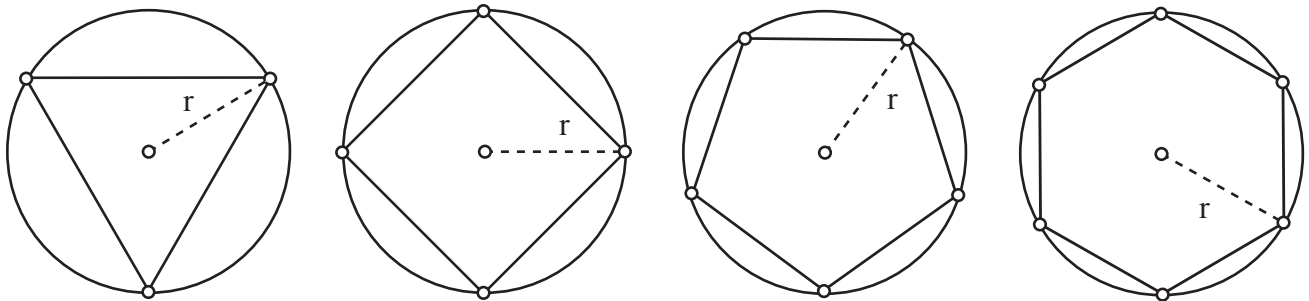
3.



The perimeter of the polygon is always less than the circumference of the circle, but the diameters for both polygon and circle are equal. This means that the ratio  $\frac{\text{perimeter}}{\text{diameter}}$  is less than the ratio  $\frac{\text{circumference}}{\text{diameter}}$ .

As the number of sides of the polygon increases, the ratio  $\frac{\text{perimeter}}{\text{diameter}}$  will get closer and closer to the value of  $\pi$ . It will never result in the exact value for  $\pi$  because no matter how many sides the inscribed polygon has, its perimeter will always be less than the circumference of the circle with the same center and radius.

*Problem 5*



Each inscribed regular polygon can be considered an approximation of the circle it is inscribed within. Consider the diameter of a polygon to be twice the radius of the inscribed circle. Therefore, the ratio of the polygon's perimeter to its "diameter" can be considered an approximation of the ratio of the circle's circumference to its diameter,  $\pi$ .

As the number of sides in the regular polygon increases, the ratio  $\frac{\text{perimeter}}{\text{diameter}}$  gets closer and closer to the value of  $\pi$ . The method yields successively more accurate approximations of  $\pi$ .

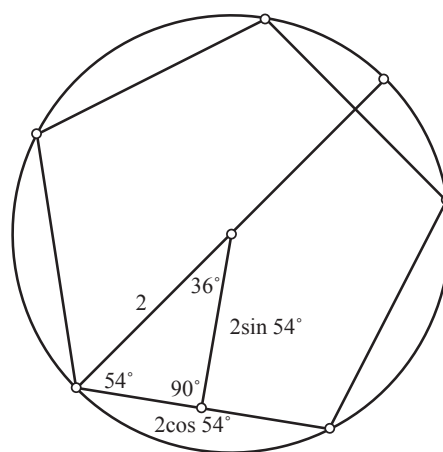
**Extension Questions:**

- Based on the chart in Problem 4, between what two values should an estimate of  $\pi$ , based on a pentagon with radius 2, fall?

The value should be between the values for a quadrilateral and a hexagon, or between 2.8284 and 3.000.

- Use construction tools or geometry software to construct a regular pentagon inscribed in a circle of radius 2. Then calculate an estimate for  $\pi$  based on the inscribed pentagon.

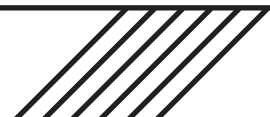
Geometry software was used to construct the figure.



Use trigonometry to solve this problem.

$$\frac{\text{perimeter}}{\text{diameter}} = \frac{20 \cos 54}{4} = 5 \cos 54 \approx 2.9398$$

The approximation for  $\pi$  is 2.9398.



### Student Work Sample

The students in this class were allowed to report their findings in a variety of ways. This student's work satisfies many of the criteria on the solution guide.

For example:

- Uses geometric and other mathematical principles to justify the reasoning and analyze the problem.

*The student uses his prior knowledge to explain his procedures.*

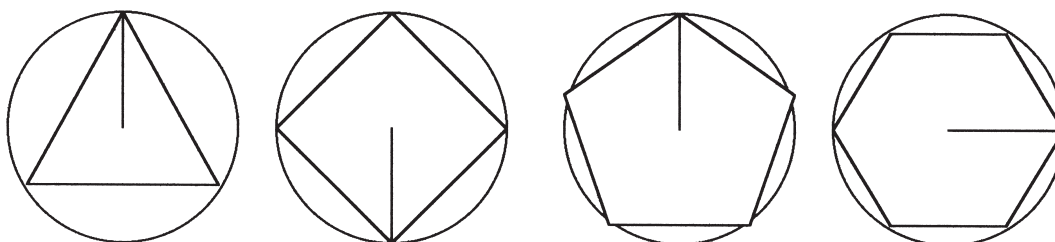
- Communicates a clear, detailed, and organized solution strategy.

*The student's narrative explains his strategy for determining the perimeter of each triangle.*

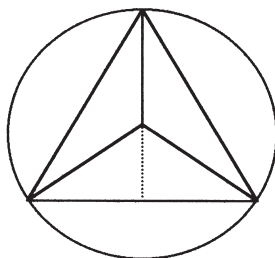
The student's work includes appropriate and accurate representations of the problem with correct diagrams that are not labeled. However, labeling may not be necessary because his narrative explains the process and the figures.

### Archimedean Walk

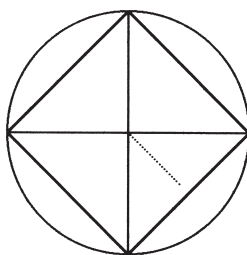
One way to find what  $\pi$  (pi) equals is by measuring the circumference of a circle and dividing it by the diameter. Archimedes realized that this would always differ by the accuracy of that person. He wanted to find away to fix the value of  $\pi$  that was based on calculations and not on measurement. He decided to put regular polygons inside circles, figure out their perimeters, and find  $\pi$  according to that shape. Archimedes decided to find  $\pi$  this way because the more sides the polygon had the closer he would get to finding the actual value of  $\pi$ .



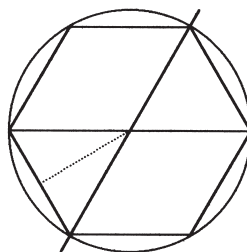
This diagram shows that the closer you get to the circumference of a circle, when you divide by the diameter, which is two, you can get closer and closer to the value for  $\pi$ . To answer the question about finding the approximation for the first, second, and fourth shape, I used trigonometry equations and special triangles. When you dissect the figures as follows you can use special triangles to solve.



When you divide the equilateral triangle in half (dotted line) you get a right triangle and a hypotenuse of two, also the radius of the circle. When we divided the triangle in half that made the central angle, which we found out using the equation to find the value for each angle in a regular polygon, divide in half also to equal  $60^\circ$ . With this knowledge you can conclude that the last angle is  $30^\circ$  and that the triangle is a 30:60:90 triangle. Since this is a special triangle we can figure out the lengths of the two legs. Knowing that the hypotenuse in two, we know that the short leg is half of that or one, and since the short leg is one, the long leg is  $1\sqrt{3}$  or just  $\sqrt{3}$ . That leaves us with half of the original triangle, and with this information we know the whole side of the triangle is  $2\sqrt{3}$ . To find the perimeter of the triangle we just multiply the value for the side times 3 to get  $6\sqrt{3}$  or  $\approx 10.39$  units. We then take the perimeter and divide it by the diameter of the circle to find the approximation of  $\pi$ , which is  $\approx 2.598$  units. This is the way in which to solve the other polygons hereafter.



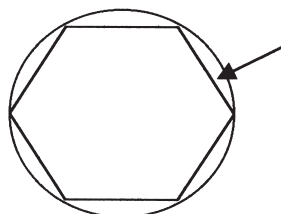
Although the triangle made a 30:60:90 triangle, the square makes a 45:45:90 triangle because when the central angle is divided by two the angle becomes a  $45^\circ$ , which makes the other a  $45^\circ$  angle also. With a 45:45:90 triangle we know that the legs are going to be the same. To find the legs we are going to divide the hypotenuse by  $\sqrt{2}$  because the hypotenuse is a leg times the  $\sqrt{2}$ , but since we already have the hypotenuse, we can reverse the equation. Using the equation,  $2/\sqrt{2}$ , we know that the two legs of the triangle are  $\approx 1.41$  units. This means that one side of the square equals  $\approx 2.23$  units. With this information we know that the perimeter of the square is  $\approx 11.31$  and that divided by 4, the diameter, equals, for the approximation of  $\pi$ ,  $\approx 2.83$  units.



The last shape, a hexagon, makes a 30:60:90 triangle when dissected. Since the angles and hypotenuse are the same as the triangles, we know that the short leg is one. That means that the length of one complete side is two units. When we multiply that by six for the perimeter, we get twelve. Then twelve divided by the diameter, four, equals three units.

# of sides of the inscribed shape	Measure of the central angle	Perimeter	Diameter	Approximation for $\pi$
3	$120^\circ$	$\approx 10.39$ units	4	2.598
4	$90^\circ$	$\approx 11.31$ units	4	2.83
6	$60^\circ$	12 units	4	3

In conclusion the more sides you add the closer the shape get to the circumference of the circle and the closer you get to the circle, when you divide by the diameter, the closer you get to the real value of  $\pi$ . This method will always be an underestimate for the value of  $\pi$ . Since you can never make an exact circle, you can not overestimate the answer. No matter how many sides you add, you can never make a circle, on reason because the sides curve in a circle, but it can come very close, yet never an exact number.



Lines can not curve, so it can never be exact.

## Talking the Archimedean Talk

In the previous performance task, Walking the Archimedean Walk, you modeled Archimedes’s method of inscribed polygons to arrive at increasingly accurate approximations for the value of  $\pi$ .

The table shows the results of that investigation.

Number of sides of the inscribed polygon	Measure of the central angle	Perimeter of inscribed polygon	Diameter of inscribed polygon	Approximation for $\pi$
3	$120^\circ$	$6\sqrt{3}$	4	2.5981
4	$90^\circ$	$8\sqrt{2}$	4	2.8284
6	$60^\circ$	12	4	3.000

In reality, Archimedes knew enough basic geometry to understand that he could start his method for approximating  $\pi$  with an inscribed hexagon. From there, he refined his approximation by doubling the sides of the hexagon and calculating the ratio  $\frac{\text{perimeter}}{\text{diameter}}$  for the resulting dodecagon.

1. Use construction tools, circular geoboards, circular dot paper, or appropriate geometry technology to construct a regular hexagon inscribed in a circle of radius 2 units.
2. Using this construction, double the number of sides to construct a regular dodecagon inscribed in a circle of radius 2 units.
3. Make appropriate calculations (no measuring allowed!) in order to determine the value of  $\pi$  based upon an inscribed regular dodecagon.



## Notes

### Materials:

Construction tools, circular geoboards, circular dot paper, or geometry software

### Geometry TEKS Focus:

(G.5) **Geometric patterns.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to:

(B) use numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles;

(D) identify and apply patterns from right triangles to solve meaningful problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

### Additional Geometry TEKS:

(G.1) **Geometric structure.** The student understands the structure of, and relationships within, an axiomatic system.

The student is expected to:

(B) recognize the historical development of geometric systems and know mathematics is developed for a variety of purposes; and

(G.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

## Teacher Notes

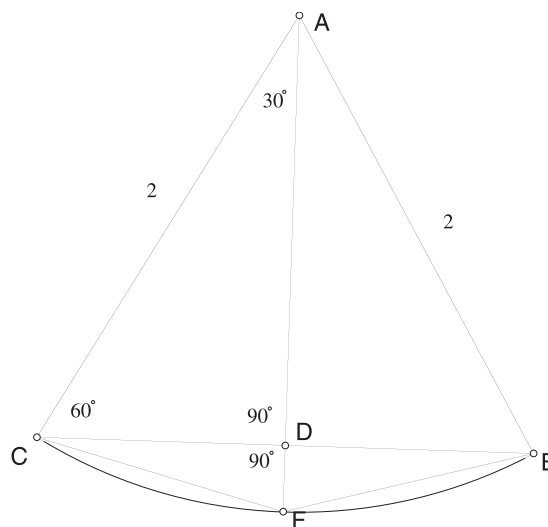
### Scaffolding Questions:

#### Problems 1 and 2

- What is the measure of a central angle of a dodecagon?
- What must the measure of the arc be that is intercepted by the dodecagon's central angle?
- What segment can you extend to intersect the circle in order to create a  $30^\circ$  arc?

#### Problem 3

Note: If necessary, have students copy the picture below.



- What are the measurements of the segments?
- What is the length of  $\overline{AF}$ ? What is the length of  $\overline{AD}$ ?
- What is the length of  $\overline{DF}$ ?

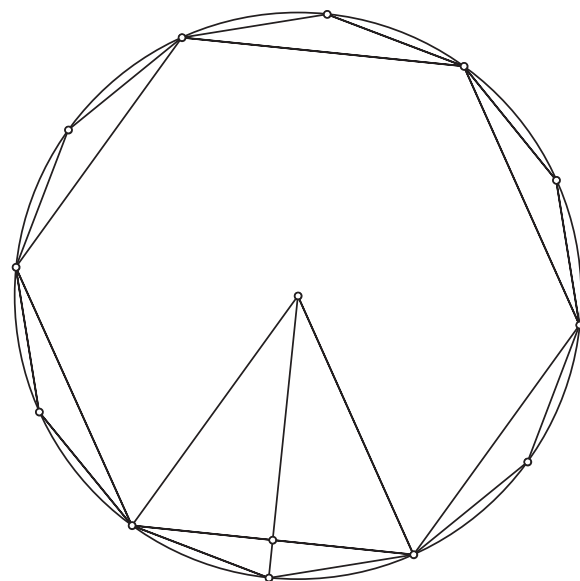
Note: If students use decimal approximations for their square roots, make sure they maintain about four decimal places of accuracy to ensure a good approximation of  $\pi$  at the final calculation.



**Sample Solutions:**

*Problems 1 and 2*

The hexagon was created by drawing a circle. With the radius of the circle as the compass width, place the compass point on the circle and strike an arc. Place the point on this mark, and strike another arc. The radius of the circle is the side of the hexagon. Connect these points to form the hexagon.



To double the sides of the hexagon, extend the hexagon's apothem until it intersects the circle. Then draw segments from the endpoints of the hexagon's sides to the point where the apothem intersects the circle. Every side of the hexagon results in two sides of the dodecagon. (Notice that this does **not** mean that the length of the dodecagon's sides is half the length of the hexagon's sides.)

Repeat this process as you go around the hexagon to create the inscribed dodecagon.

The student is expected to:

(A) use constructions to explore attributes of geometric figures and to make conjectures about geometric relationships; and

**(G.4) Geometric structure.**

The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to select an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

**(G.5) Geometric patterns.**

The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to:

(A) uses numeric and geometric patterns to develop algebraic expressions representing geometric properties;

**(G.8) Congruence and the geometry of size.**

The student uses tools to determine measurements of geometric figures and extends measurement concepts to find perimeter, area, and volume in problem situations.

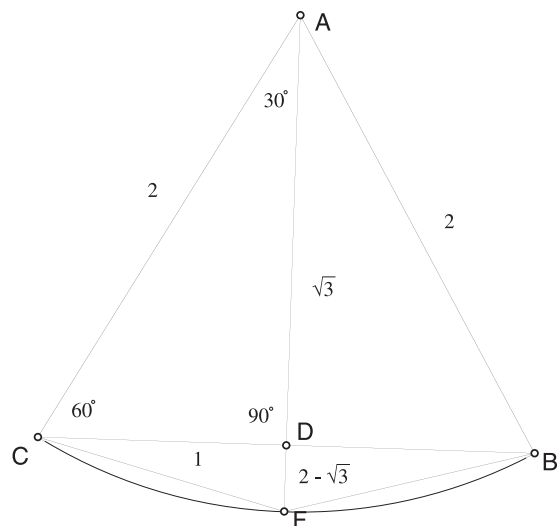
The student is expected to:

(C) derive, extend, and use the Pythagorean Theorem; and

**Connections to TAKS:**

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Problem 3



CF, the length of the side of the dodecagon, needs to be found in order to complete the calculation for  $\pi$ .

In right  $\triangle ADC$ :

$AC = 2$ ;  $CD$  is one-half of  $AC$ , or 1 unit, and  $AD$  is  $\sqrt{3}$ .

The length of  $\overline{AF}$  is 2, so  $DF = AF - AD$  or  $2 - \sqrt{3}$ .

Use the Pythagorean Theorem on right  $\triangle DCF$ .

Use  $\sqrt{3} \approx 1.7321$

$$1^2 + (2 - 1.7321)^2 \approx CF^2$$

$$1 + (0.2679)^2 \approx CF^2$$

$$1 + 0.0718 \approx CF^2$$

$$\sqrt{1.0718} \approx CF$$

$$1.0353 \approx CF$$

The perimeter of the regular dodecagon is approximately  $12(1.0353)$ , or 12.4236 units.

So for the dodecagon, ratio  $\frac{\text{perimeter}}{\text{diameter}} \approx \frac{12.4236}{4} \approx 3.1059$ .

Using exact values:

$$1^2 + (2 - \sqrt{3})^2 = CF^2$$

$$1 + 4 - 4\sqrt{3} + 3 = CF^2$$

$$8 - 4\sqrt{3} = CF^2$$

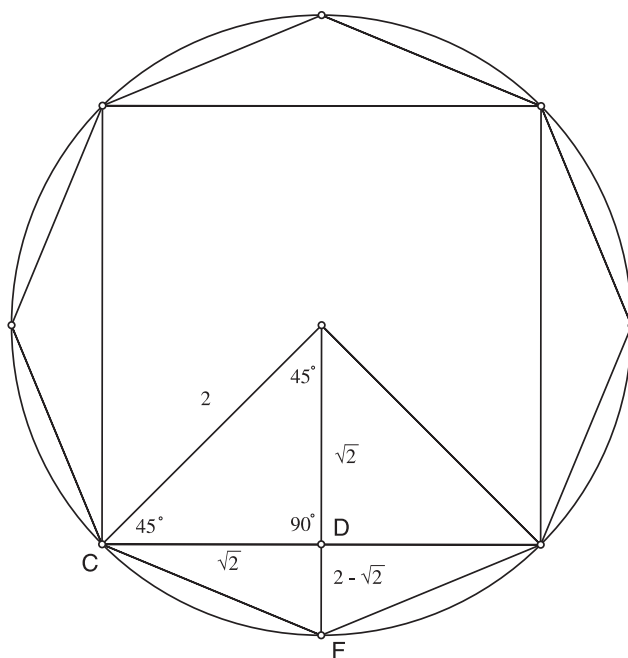
$$4(2 - \sqrt{3}) = CF^2$$

$$CF = \sqrt{4(2 - \sqrt{3})} = 2\sqrt{(2 - \sqrt{3})}$$

So for the dodecagon, ratio  $\frac{\text{perimeter}}{\text{diameter}} \approx \frac{24\sqrt{2 - \sqrt{3}}}{4} = 6\sqrt{2 - \sqrt{3}} \approx 3.1058$ .

**Extension Questions:**

- Between what two values should an estimate of  $\pi$  based on an octagon fall?  
The value should be between the values for a hexagon and a dodecagon, or between 3.000 and 3.1058.
- Calculate an estimate for  $\pi$  based on an inscribed octagon.



Construct a square and extend the apothem to intersect the circle at point F. Connect this point to the adjacent vertices of the square. Repeat this process for each apothem. The  $\overline{DF}$  measures  $2 - \sqrt{2}$ .

In right  $\triangle CDF$ :

$$\text{Using } \sqrt{2} \approx 1.4142$$

$$1.4142^2 + (2 - 1.4142)^2 \approx CF^2$$

$$2 + (0.5858)^2 \approx CF^2$$

$$2 + 0.3432 \approx CF^2$$

$$\sqrt{2.3432} \approx CF$$

$$1.5308 \approx CF$$

So for the octagon,  $\frac{\text{perimeter}}{\text{diameter}} \approx \frac{12.2464}{4} \approx 3.0616$ .

This value is between the values for a hexagon and a dodecagon.

Using exact values:

$$(\sqrt{2})^2 + (2 - \sqrt{2})^2 = CF^2$$

$$2 + 4 - 4\sqrt{2} + 2 = CF^2$$

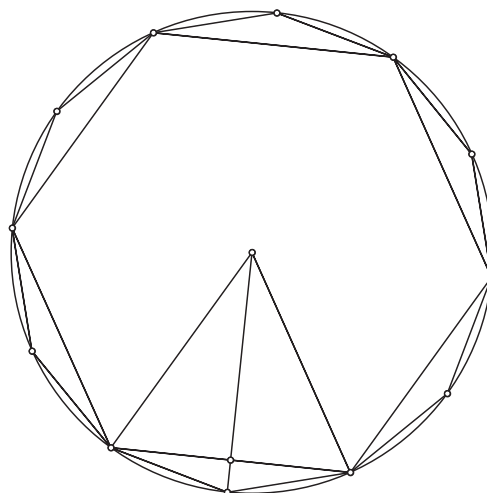
$$8 - 4\sqrt{2} = CF^2$$

$$4(2 - \sqrt{2}) = CF^2$$

$$CF = \sqrt{4(2 - \sqrt{2})} = 2\sqrt{(2 - \sqrt{2})}$$

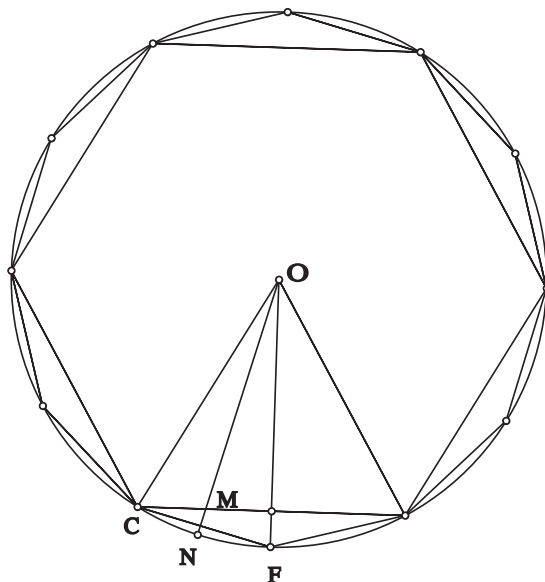
So for the octagon, ratio  $\frac{\text{perimeter}}{\text{diameter}} \approx \frac{16\sqrt{2 - \sqrt{2}}}{4} = 4\sqrt{2 - \sqrt{2}} \approx 3.0615$ .

- Double the sides of a dodecagon. Then calculate an estimate for  $\pi$  based on an inscribed 24-gon. (Circle with radius 2)



To double the sides of the hexagon, extend the hexagon's apothem until it intersects the circle. Then draw segments from the endpoints of the hexagon's sides to the point where the apothem intersects the circle. Every side of the hexagon results in two sides of the dodecagon. (Notice that this does **not** mean that the length of the dodecagon's sides is half the length of the hexagon's sides.)

Repeat this process as you go around the hexagon to create the inscribed dodecagon.



$$CF = 1.0353$$

$$CM = \frac{1}{2}(1.0353)$$

$$CO = 2$$

$$OM^2 + \left(\frac{1}{2}(1.0353)\right)^2 = 2^2$$

$$OM \approx 1.932$$

$$MN = 2 - OM \approx 0.06815$$

$$CN^2 = MN^2 + CM^2$$

$$CN^2 = (0.06815)^2 + \left(\frac{1}{2}(1.0353)\right)^2$$

$$CN = 0.522$$

$$\text{Perimeter} = 24(0.522) = 12.528$$

$$\frac{\text{perimeter}}{\text{diameter}} \approx \frac{12.528}{4} \approx 3.132.$$

Note: Archimedes extended his calculations to an estimate based on an inscribed 96-gon.



## Mad as a Hatter or Hat as a Madder

“Then you should say what you mean,” the March Hare went on.

“I do,” Alice hastily replied, “at least—at least I mean what I say—that’s the same thing, you know.”

“Not the same thing a bit!” said the Hatter. “Why, you might just as well say that ‘I see what I eat’ is the same thing as ‘I eat what I see’!”

Lewis Carroll, *Alice in Wonderland*

### Problem Set 1

For each of the following conditional statements:

- a) Determine if the original statement is true. Explain in writing the reasoning for each of your choices.
  - b) Write the converse of the statement.
  - c) Determine if the converse of the statement is true or false. Explain in writing the reasoning for each of your choices.
1. If a number is divisible by 4, then it is divisible by 2.
  2. If it is raining in Las Vegas, then it is sunny in Nevada.
  3. If it is sunny in Nevada, then it is sunny in Las Vegas.
  4. If a number is greater than -500, then the number is greater than 500.
  5. If a person is a teenager, then that person is between 13 and 19 years old.
  6. If a musician plays classical guitar, then the musician is a female.
  7. If a female plays classical guitar, then she is a musician.
  8. If a person reads a lot, then that person is smart.
  9. If a number is even, then it is divisible by 2.
  10. If a car is an SUV, then it has four wheels.

11. If a number squared is greater than 1, then the number itself must be greater than 1.
12. If point B is the midpoint of  $\overline{AC}$ , then the distance from point A to point B is the same as the distance from point B to point C.
13. If a polygon has exactly eight sides, then it is an octagon.
14. If a parallelogram has a right angle, then it is a square.
15. If a quadrilateral has exactly one pair of parallel sides, then it is a trapezoid.

### Problem Set 2

- Write:
1. a non-mathematical statement
- and
2. a mathematical statement

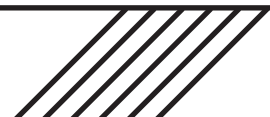
that fulfill the following requirements.

1. A true conditional statement with a converse that is also true.
2. A true conditional statement with a converse that is false.
3. A false conditional statement with a converse that is true.
4. A false conditional statement with a converse that is also false.

### Problem Set 3

1. Present your statements to fellow classmates. Be prepared to explain and justify why you consider your statements either true or false.
2. Write a few sentences discussing which types of statements were easiest to agree on and which types provoked the most disagreement and discussion.
3. Which statements in Problem Set 1 and Problem Set 2 would you consider to be definitions? Why?







## Notes

### Materials:

One pencil or pen for each student

### Geometry TEKS Focus:

(G.3) **Geometric structure.** The student applies logical reasoning to justify and prove mathematical statements.

The student is expected to:

(A) determine the validity of a conditional statement, its converse, inverse, and contrapositive;

(C) use logical reasoning to prove statements are true and find counter examples to disprove statements that are false;

## Teacher Notes

### Scaffolding Questions:

#### *Problem Set 1*

- In writing the converse of a conditional statement: “What is the hypothesis?” and “What is the conclusion?”
- Guide students as necessary toward composing correct grammatical sentences rather than simply mechanically switching the hypothesis with the conclusion.
- Guide students as necessary toward providing specific, concrete counterexamples in order to determine the converse or to establish that statements are false.

For the converse of the statement in Problem Set 1, question 1: “Can you think of a number that is divisible by 2, but that is not even?” The student response might be: “If it is divisible by two, then a number is divisible by 4.” Guiding question: “Can you rewrite your sentence so that the subject (a number) is part of the hypothesis?”

For Problem Set 1, question 10: Students will justify that the converse of this statement is false with a general statement like: “Not all cars are SUVs.” An appropriate question would then be: “Can you think of a specific car that has four wheels but is not an SUV?”

#### *Problem Sets 1, 2, and 3*

- Student examples often provoke debate among classmates about the relative merit and/or truth of counterexamples. They might not always agree that a statement is a valid counterexample.
- Students are asked to address this issue in Problem Set 3. Guide students as necessary toward making the distinction between non-mathematical statements, whose truth can depend on people’s viewpoints or opinions, and mathematical statements, whose truth depends on logical reasoning.

**Sample Solutions:**

*Problem Set 1*

1. **True.** Since 4 is divisible by 2, then numbers divisible by 4 must also be divisible by 2.

**Converse:** If a number is divisible by 2, then it is divisible by 4.

**False.** Numbers such as 2, 6, and 10 are divisible by 2, but not by 4.

2. **False.** If it is raining in Las Vegas, then it is not sunny throughout the entire state of Nevada.

**Converse:** If it is sunny in Nevada, then it is raining in Las Vegas.

**False.** Las Vegas would be sunny if it were sunny in Nevada; therefore, it can't be raining in Las Vegas.

3. **True.** Las Vegas is part of the entire sunshine-drenched state of Nevada.

**Converse:** If it is sunny in Las Vegas, then it is sunny in Nevada.

**False.** It could be raining in Reno, Nevada, while it is sunny in Las Vegas.

4. **False.** 300 is greater than -500 but not greater than 500.

**Converse:** If a number is greater than 500, then it is greater than -500.

**True.** All numbers greater than 500 are positive, and by definition all positive numbers must be greater than negative numbers like -500.

5. **True.** This is the definition of a teenager.

**Converse:** If a person is between 13 and 19 years old, then that person is a teenager.

**True.** This is the definition of a teenager.

6. **False.** For example, Segovia was a famous male classical guitarist.

**Converse:** If a musician is a female, then the musician plays classical guitar.

**Additional Geometry TEKS:**

(G.1) **Geometric structure.**

The student understands the structure of, and relationships within, an axiomatic system.

The student is expected to:

(A) develop an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems;

**Connections to TAKS:**

NONE

**False.** The Dixie Chicks do not play classical guitar, but they are female musicians. (This is, of course, open to debate. Like many non-mathematical statements, we can't always be perfectly sure if they are true or false.)

7. **True.** Classical guitar players are all musicians.

**Converse:** If a female is a musician, then she plays classical guitar.

**False.** Gloria Estefan, a female musician, is a percussionist.

8. **False.** Students can argue that there are many ways to have intelligence which have nothing to do with reading.

**Converse:** If a person is smart, then they read a lot.

**False.** This is false for the same reason offered above. (This is another example of a non-mathematical statement whose truth is open to debate and interpretation.)

9. **True.** There are no even numbers that are not divisible by 2.

**Converse:** If a number is divisible by 2, then it is even.

**True.** There are no numbers that are even that are not divisible by 2. Or, this is true by definition of an even number.

10. **True.** All cars have four wheels and an SUV is a type of car.

**Converse:** If a car has four wheels, then it is an SUV.

**False.** A Volkswagen Beetle is not an SUV, and it is a car with four wheels.

11. **False.** A negative number is less than 1 even though its square might be greater than 1.

**Converse.** If a number is greater than 1, then the number squared is greater than 1.

**True.** Squaring a number greater than 1 makes the number even larger.

12. **True.** If B is the midpoint of  $\overline{AC}$ , then by definition,  $\overline{AB} = \overline{BC}$ .

**Converse:** If the distance from point A to point B is the same as the distance from point B to point C, then B is the midpoint of  $\overline{AC}$ .

**False.**  $\overline{AB}$  and  $\overline{BC}$  can be the legs of an isosceles triangle.

13. **True.** This is the definition of an octagon.

**Converse:** If a polygon is an octagon, then it has eight sides.

**True.** By definition, an eight-sided polygon is an octagon.

14. **False.** It could be a rectangle.

**Converse:** If a parallelogram is a square, then it has a right angle.

**True.** All squares have right angles by definition.

15. **True.** This is the definition of a trapezoid.

**Converse:** If a quadrilateral is a trapezoid, then it has exactly one pair of parallel sides.

**True.** This is the definition of a trapezoid.

### *Problem Set 2*

Answers will vary. Check student work.

1. Question 5, Problem Set 1 is a true conditional with a true converse.
2. Question 1, Problem Set 1 is a true conditional with a false converse.
3. Question 14, Problem Set 1 is a false conditional with a true converse.
4. Question 2, Problem Set 2 is a false conditional with a false converse.

### *Problem Set 3*

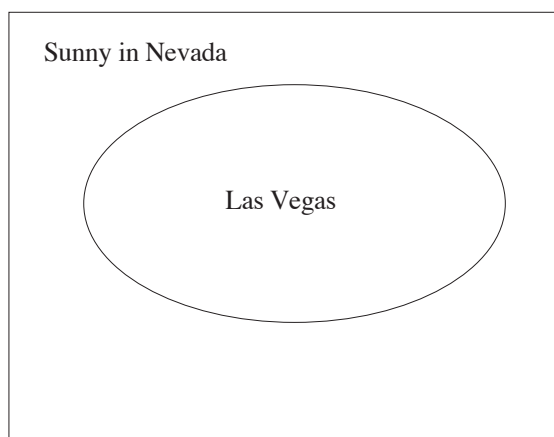
- 1–2. Answers will vary. Students should be able to articulate that, in general, it's easier to decide the truth/falseness of mathematical statements than non-mathematical statements. The terms in mathematical statements are precisely defined, and their truth depends on logic. In contrast, the truth/falseness of non-mathematical statements often depends on people's opinions and personal understanding of what words mean.
3. The statements that make good definitions are the conditional statements that have true converses.

**Extension Questions:**

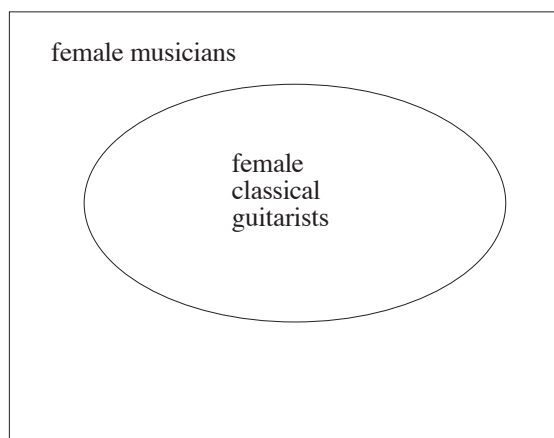
- Refer to questions 3, 7, and 10 in Problem Set 1. For each of these questions, represent the conditional statement in an Euler Diagram.

Note: Euler (pronounced “oiler”) diagrams are often called Venn diagrams.

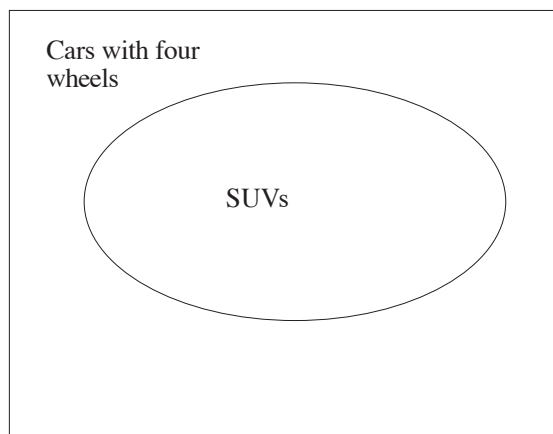
*Problem 3*



*Problem 7*



## Problem 10



- Write definitions of the following terms as true conditional statements with true converses. Then rewrite the definitions as biconditional statements.
  - a) complementary angles
  - b) isosceles triangle
  - c) polygon
  - a) *If two angles are complementary, then the sum of their measures is  $90^\circ$ . Two angles are complementary if and only if the sum of their measures is  $90^\circ$ .*
  - b) *If a triangle is isosceles, then it has at least two sides that are congruent. A triangle is isosceles if and only if it has at least two sides that are congruent.*
  - c) *If a shape is a polygon, then it is a plane figure formed from three or more line segments, such that each segment intersects exactly two other segments, one at each endpoint, and no two segments with a common endpoint are collinear.*  
*A shape is a polygon if and only if it is a plane figure formed from three or more line segments, such that each segment intersects exactly two other segments, one at each endpoint, and no two segments with a common endpoint are collinear.*
- Create and name your own object. It does not have to be mathematical, but it should be something you can draw. Then, write the definition of your object as a biconditional statement.

*Answers will vary. Check student work.*





## Going the Distance in Taxicab Land (Assessment)

Write a paragraph comparing and contrasting the characteristics of the geometric objects you studied in the Going the Distance in Taxicab Land lesson as they are represented on the traditional coordinate grid and as they are represented on a taxicab coordinate grid.

1. Be sure to include all of the following objects in your analysis:

line segments

circles

perpendicular bisectors

2. Discuss whether you think the definitions or characteristics of these objects are valid and useful for both geometric systems.

3. Can you think of situations that might be better represented in taxicab geometry than in traditional geometry?

**Going the Distance in Taxicab Land (Lesson):  
Describing and Analyzing Objects in Two Different Geometric Systems.**

*Lesson Activity 1:*

Draw accurate sketches to represent the bold face definitions, theorems, or postulates, and answer the questions that follow.

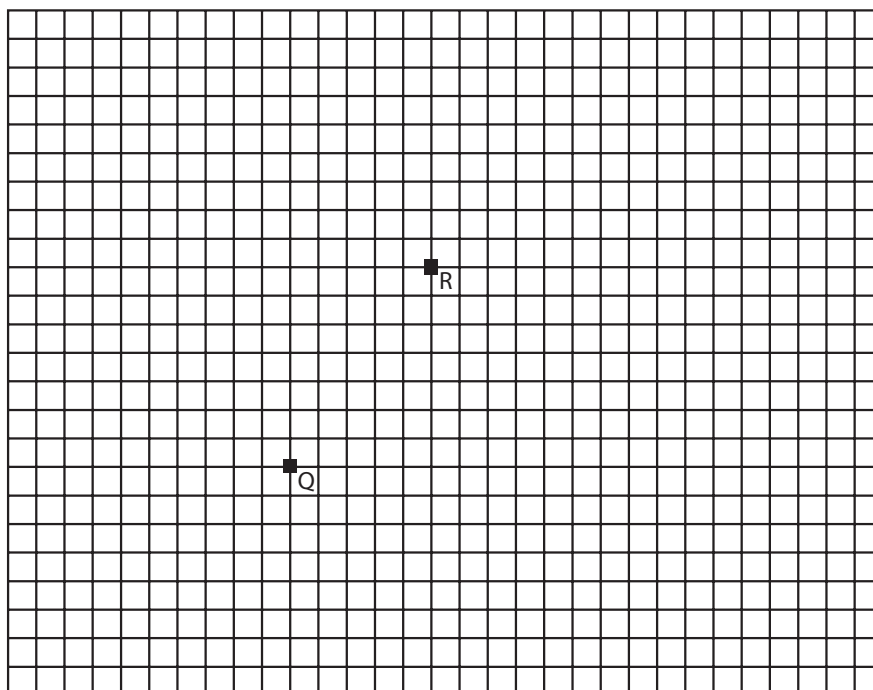
**1. Two points determine a line segment.**

• R

• Q

- a) How many line segments can be drawn between the two given points?
- b) The line segment drawn between the two points represents the \_\_\_\_\_ distance between the two points.

**2. Make the same sketch on a coordinate grid.**



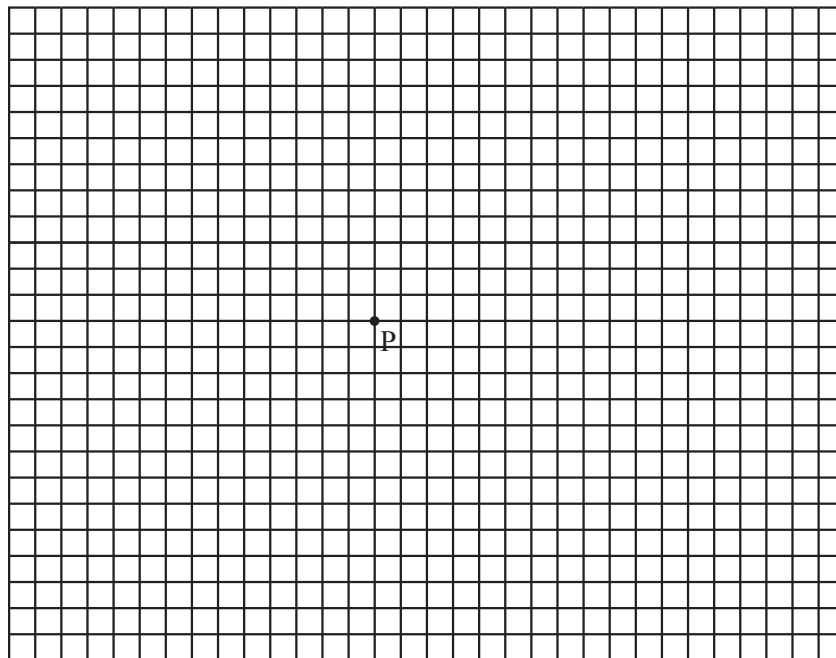
- a) How many line segments can be drawn between the two points?
- b) The line segment drawn between the two points represents the \_\_\_\_\_ distance between the two points.
- c) Does placing the geometric object on the coordinate grid change its characteristics? Explain your answer.
3. **A circle is the set of all points in a plane that are the same distance from a given point in the plane.**

Sketch a circle with center at point P and a radius of 6 cm.

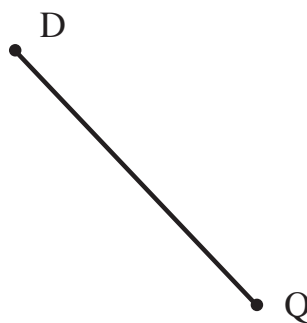


- a) How many different circles can you draw with this center and radius?

4. Make the same sketch on a coordinate grid. (For convenience, make the radius of the circle 6 grid units.)

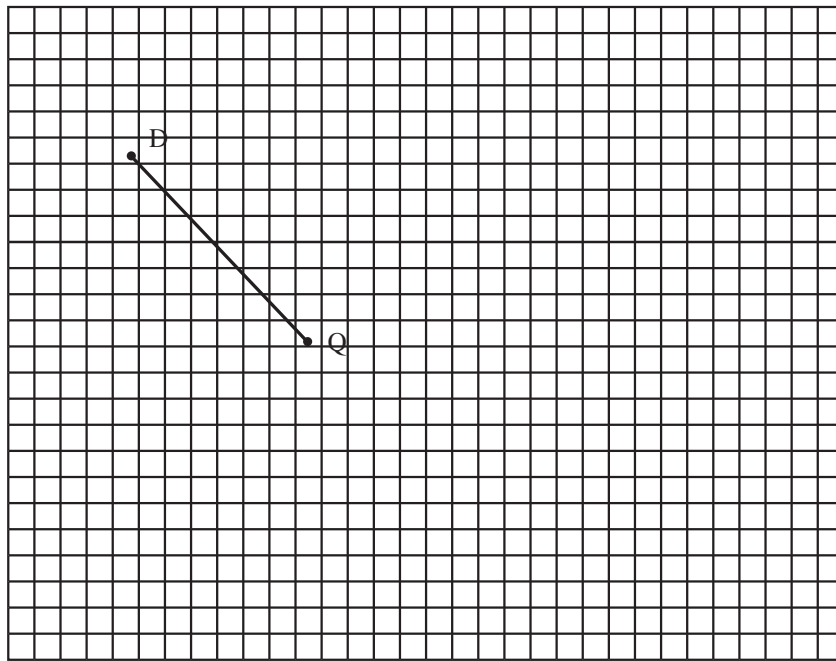


- a) How many different circles can you draw with this center and radius?
  - b) Does placing the geometric object on the coordinate grid change its characteristics? Why?
  - c) Does changing the location of the geometric object on the coordinate grid change its characteristics?
5. **A point is on a segment's perpendicular bisector if and only if it is the same distance from each of the segment's endpoints.** (Represent all the points that satisfy this requirement.)



How many different perpendicular bisectors is it possible to draw for the given line segment?

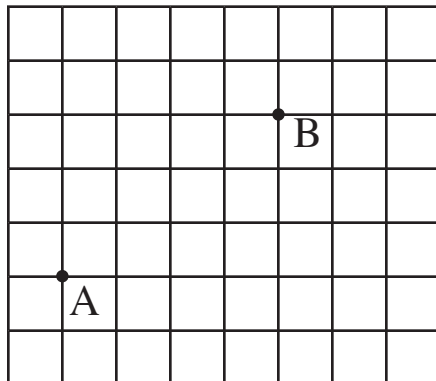
6. Make the same sketch on a coordinate grid.



- How many different perpendicular bisectors is it possible to draw for the given line segment?
- Does placing the geometric object on the coordinate grid change its characteristics? Why or why not?
- Describe how changing the position of the geometric object on the coordinate grid changes its characteristics.

*Lesson Activity 2:*

1. Determine the distance between points A and B on the coordinate grid below.



2. Imagine you are a taxicab driver and the coordinate grid above represents the grid of city streets you can travel on. Determine the **taxidistance** from point A to point B in Taxicab Land.
3. In what ways do you think the coordinate grid in Taxicab Land is different from the traditional coordinate grid you worked with in Activity 1?

*Lesson Activity 3:*

You have seen that representing a geometric object on a traditional coordinate grid does not change any of its characteristics. Neither does repositioning the object on the grid.

Your task is to analyze what happens when geometric objects are placed on a taxicab grid which has the following characteristics:

Points on a taxicab grid can be located only at the intersections of horizontal and vertical lines.

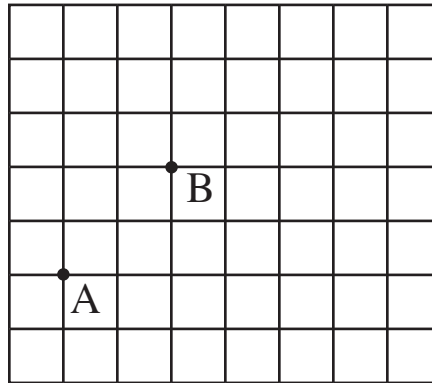
One unit will be one grid unit.

The numerical coordinates of points in **taxicab geometry** must therefore always be integers.

The **taxidistance** between two points is the smallest number of grid units that an imaginary taxi must travel to get from one point to the other. In Activity 2, the **taxidistance** between point A and point B is 7.

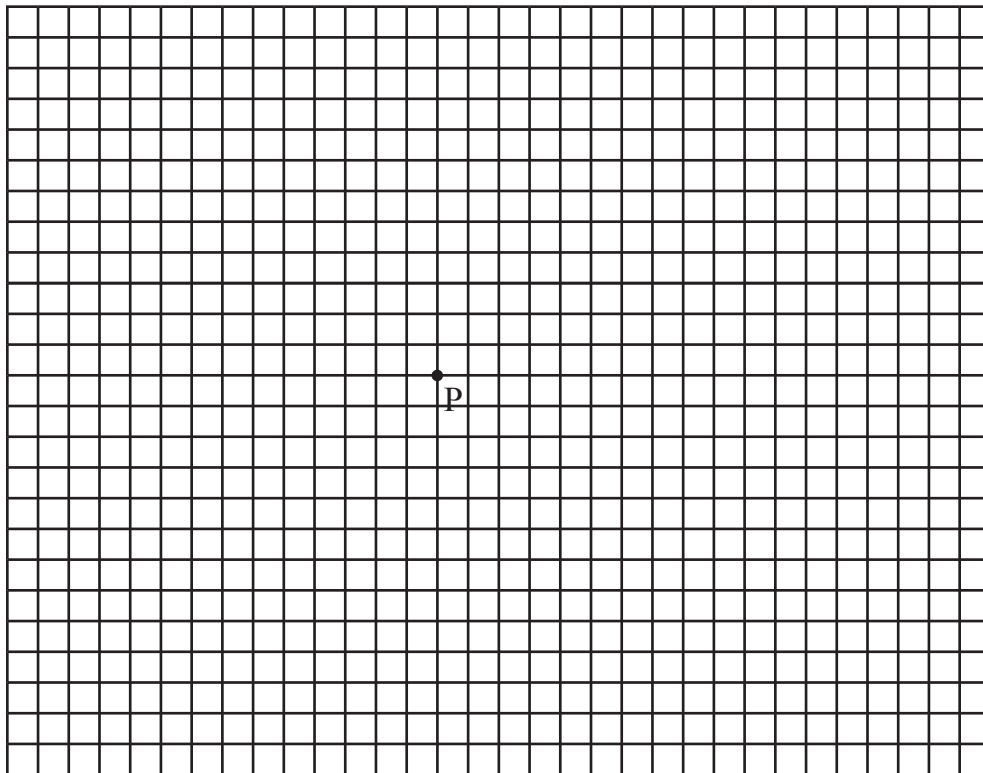
Draw accurate sketches on a **taxicab geometry** coordinate grid to represent the following definitions, theorems, or postulates, and answer the questions that follow.

**1. Two points determine a line segment.**



- a) Is this the only line segment that can be drawn between the two given points? Explain.
- b) The line segment(s) drawn between the two points represents the \_\_\_\_\_ taxidistance between the two points.
- c) Does changing the position of the geometric object on the taxicab grid change its characteristics? Explain.

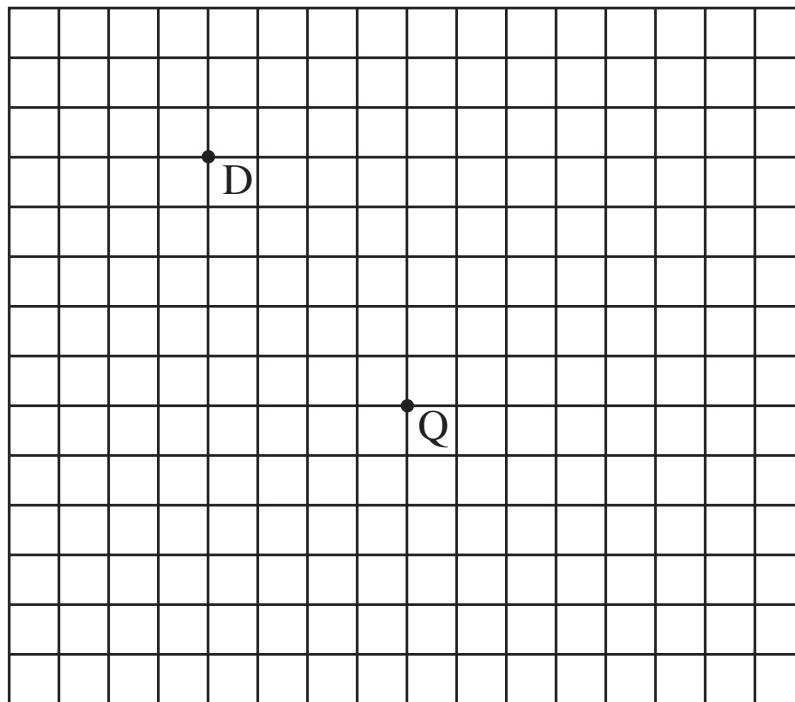
2. **A circle is the set of all points in a plane that are the same distance from a given point in the plane.** (Sketch a circle with a radius of 6.)



- a) How many different circles can you draw with this center and radius?  
Explain your answer.
- b) Explain how changing the position of the geometric object on the taxicab grid changes its characteristics.



3. **A point is on a segment's perpendicular bisector if and only if it is the same distance from each of the segment's endpoints.** (Represent all the points that satisfy this requirement.)



- a) Is the set of points you represented in your drawing the only perpendicular bisector for the two given endpoints? Explain.
- b) Does changing the position of the geometric object on the coordinate grid change its characteristics? Explain.



## Notes

### Materials:

One compass and ruler per student

### Geometry TEKS Focus:

#### (G.1) Geometric structure.

The student understands the structure of, and relationships within, an axiomatic system.

The student is expected to:

(A) develop an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems;

(C) compare and contrast the structures and implications of Euclidean and non-Euclidean geometries.

### Additional Geometry TEKS:

#### (G.7) Dimensionality and the geometry of location.

The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student is expected to:

(A) use one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures;

## Teacher Notes

The lesson is intended to allow students to develop “an awareness of the structure of a mathematical system” (see TEK b.1 (A)). If students are already familiar with taxicab geometry, the assessment may be used to evaluate their understanding of this mathematical system.

### Scaffolding Questions:

#### Assessment:

Encourage students to review all the information in the lessons. They may want to organize the information in a chart form before they write their summary paragraph.

#### Lesson Activity 1:

1. What does the word “determine” mean in the sentence: “Two points determine a line segment”?
2. Is the coordinate grid line segment drawn between point Q and R unique? Does it represent the shortest distance between those two points?
4. Students may notice that point P does not lie at a grid intersection.
  - Is it possible to draw a circle with a radius of 6 with point P positioned this way?
  - After students draw the initial circle, they may position point P at a grid point and redraw the circle.
6. Students may notice that the segment does not have endpoints that lie on grid intersections.
  - Is it possible to draw the perpendicular bisector with the points in their present position?
  - Students may reposition the segment so that its endpoints lie on grid intersections and then complete the sketch.

*Lesson Activity 2:*

1. Is it possible to find the distance between A and B by counting along the grid lines?

What formula could we use to determine the distance between A and B?

2. In Taxicab Land is it possible to find the distance between A and B by counting along the grid lines?

Is there more than one way to count out the distance?

Which way (or ways) of counting out the distance do you think is correct?

3. Where can points be located on a traditional coordinate grid?

Where can points be located on a taxicab coordinate grid?

What does this tell us about the numerical coordinates of the points on each type of coordinate grid?

How is the distance between two points on a taxicab coordinate grid different from the distance on a traditional coordinate grid?

*Lesson Activity 3:*

1. Students may notice that there are many ways to draw a minimum distance pathway between the two points.

Do all these segments represent the shortest taxidistance between the two points?

2. Do the points you drew satisfy the definition of a circle?

Should you connect the points by drawing along the grid lines so that it looks more like an enclosed shape?

3. Do you think there will be any points that are the same distance from both endpoints?

**Connections to TAKS:**

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

**One student says in response to Question 2 . . .**

“In normal geometry, a line is the shortest distance between two points. In taxicab it is not. In normal geometry, circles have points in which the shortest distance from the center to each point is equal. In taxicab geometry, the circles are different. They are not curved. In normal geometry, all segments have a perpendicular bisector. In taxicab geometry only segments with an even taxicab distance have perpendicular bisectors. The definitions for geometric shapes are only useful in normal geometry, not taxicab. You would use taxicab geometry if you wanted to find out how far a car had to travel to get from one street corner to another.”

**Sample Solutions:***Assessment:***Traditional Coordinate System**

Line Segment:	Uniquely determined by two points. Represents shortest distance between two points.
Circle:	Uniquely determined by given center and radius. Characteristics do not change when repositioned.
Perpendicular Bisector:	Uniquely determined by endpoints of segment. Characteristics do not change when repositioned.

**Taxicab Coordinate System**

Line segment:	Not necessarily uniquely determined by two points. Represents shortest distance between two points.
Circle:	Uniquely determined by center and radius. Characteristics do not change when repositioned.
Perpendicular Bisector:	May not exist at all. If it does exist, however, then it is uniquely determined by endpoints of segment.  Characteristics change when repositioned.

Students should realize that the definitions and characteristics may be more useful and familiar in the traditional system, but they are equally valid in both systems.

Any sort of problem where one needs to find locations along a grid of city streets might be more successfully represented using taxicab geometry.

*Lesson Activity 1:*

1. a) One  
b) The line segment drawn between the two points represents the shortest distance between the two points.
2. a) One  
b) The line segment drawn between the two points represents the shortest distance between the two points.  
c) No. There is one and only one line segment between the two points, and it represents the shortest distance between them.
3. Draw the circle with a compass. Only one circle may be drawn with a given radius.
4. A circle can be drawn on a coordinate grid with a radius of 6 grid units by using a compass.

If students subsequently reposition point P on a grid point, then they can sketch the circle by counting grid units, make a ruler in grid units, etc.

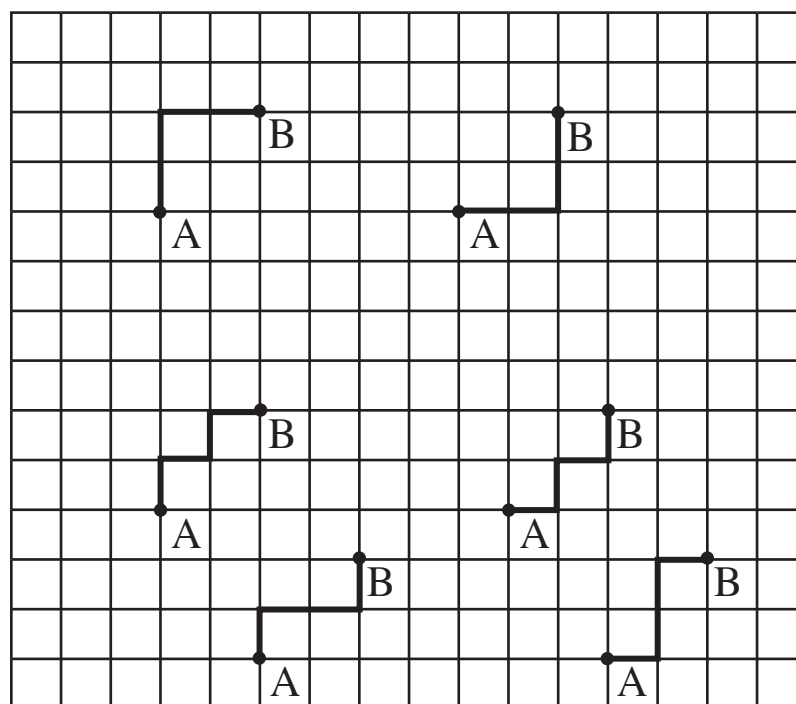
- a) One  
b) No. There is only one circle with the given radius and center.  
c) No. The coordinate name of the point changes but the properties of the circle stay the same.
5. Compass and straightedge construction is the most accurate way to do this. One perpendicular bisector can be drawn for the given line segment.
6. Compass and straightedge construction is still an appropriate way to do this. If students reposition points D and Q to the nearest grid intersections, then the 10-unit segment's midpoint will also fall on a grid intersection.  
a) One  
b) No. There is only one set of points that satisfy the requirement of being the same distance from the endpoints of segment  $\overline{DQ}$ .  
c) No

*Lesson Activity 2:*

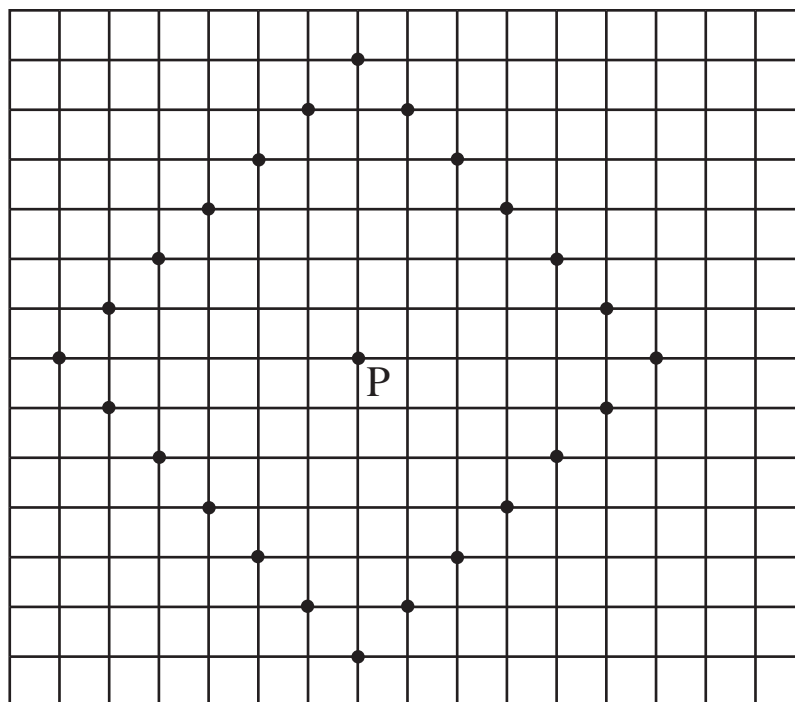
1.  $\overline{AB}$  is the hypotenuse of a right triangle with sides of length 3 units and 4 units. By the Pythagorean Theorem, the length of  $\overline{AB}$  is 5 units.
2. The taxi driver must travel on the streets, so he would travel 3 blocks vertically and 4 blocks horizontally. He would travel a total distance of 7 blocks.
3. Points on a taxicab grid can be located only at grid intersections. This makes their numerical coordinates integers. The taxidistance between two points must always be an integer.

*Lesson Activity 3:*

1. a) There are six minimum distance pathways that can be drawn between the two points.

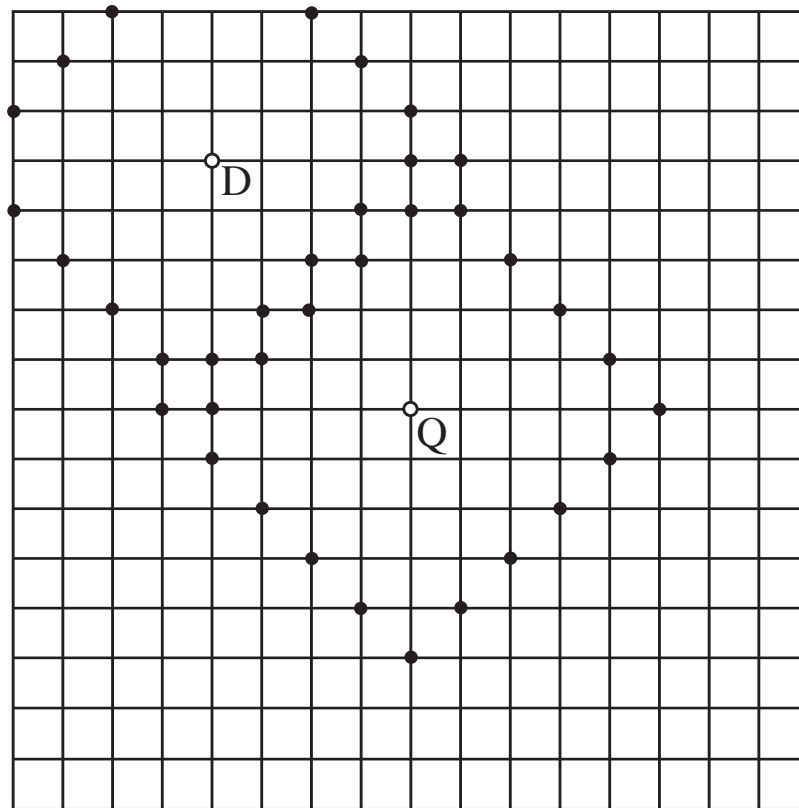


- b) The line segment(s) drawn between the two points represent the shortest taxidistance between the two points.
- c) Yes. Students should realize that if they position the two points along either a horizontal or a vertical grid line, then there will be only one minimum distance segment between them.



2. a) There is only one set of points that are all 6 units from P. They are points that lie on “street corners,” such that the sum of the horizontal and vertical distances is 6 units.
- b) The coordinates of the points would change, but the shape of the figure would remain the same.
3. a) There are no points that satisfy the requirement of being the same distance from points D and Q. The distance from point D to point Q is 9 units.

The graph shows the set of points that are 5 units from D and the set of shaded points that are 5 units from Q. There are no common points. The two taxicab circles will not intersect.

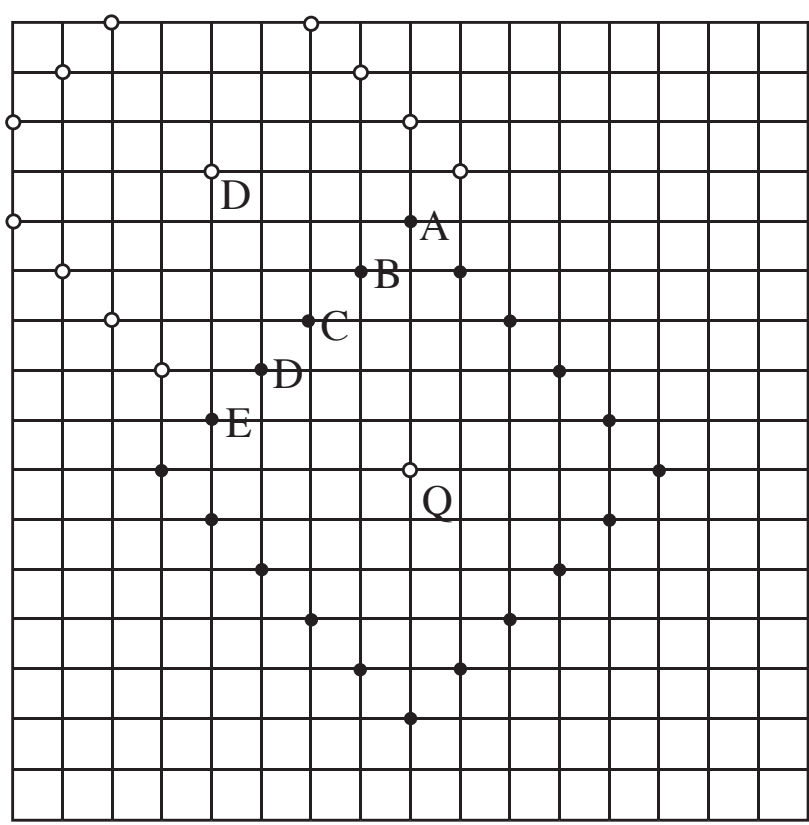


- b) Students may experiment and discover that if the taxidistance between the two points is odd, then there will be no perpendicular bisector.

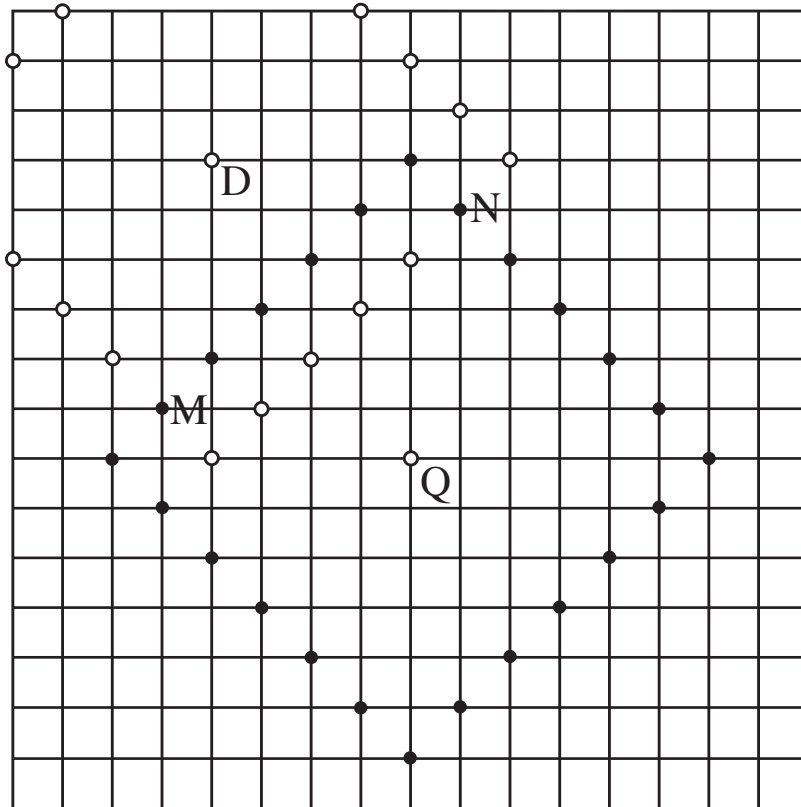
If, however, the taxidistance between the two points is even, there will be a perpendicular bisector, and it can take on a variety of configurations, depending on how the points are positioned.



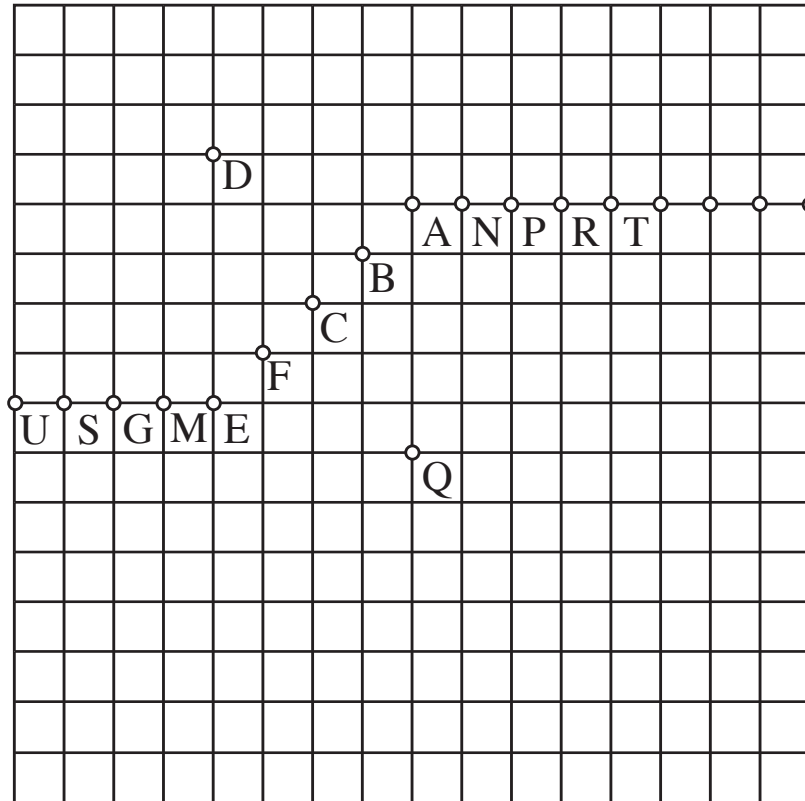
D and Q in this graph are 10 units apart. The next graph shows the two circles that have a radius of 5 units and centers D and Q. The points that the two circles have in common (A, B, C, D, and E) are all 5 units from both D and Q. Then A, B, C, D, and E are points on the perpendicular bisector of D and Q.



The next graph shows the two circles that have a radius of 6 units and centers D and Q.  
The points that the two circles have in common, M and N, are 6 units from both D and Q.



This next graph shows the points A, B, C, D, and E (5 units from D and Q), M and N (six units from D and Q), P and Q (seven units from D and Q), R and S (eight units from D and Q), T and U (nine units from D and Q). This collection of points is a part of the set of points equidistant from D and Q.



**Extension Questions:**

- For each pair of points given, decide if it is possible to draw a perpendicular bisector in a taxicab coordinate system.

(1, 3) and (5, 3)                      yes

(-2, 0) and (-6, -4)                  yes

(5, -2) and (2, 1)                      yes

(-3, 3) and (2, 1)                      no

(5, -2) and (-3, 3)                    no

(5, 3) and (5, -1)                      yes

- Write a conjecture about what must be true in order for it to be possible to draw a perpendicular bisector in a taxicab coordinate system.

The taxidistance between the two endpoints must be an even number.

- Experiment with circles of varying radii on a taxicab coordinate system. Write a conjecture about the value of  $\pi$  in taxicab geometry.

The value of  $\pi$  in taxicab geometry is 4.

**Chapter 3:**  
*Properties and  
Relationships of  
Geometric Figures*



# Introduction

The set of performance tasks in Chapter 3 requires students to analyze properties and describe relationships in geometric figures, including parallel and perpendicular lines, circles and lines that intersect them, and polygons and their angles. The students will use explorations to formulate and test conjectures.





## Angle Bisectors and Parallel Lines

Construct two parallel lines. Draw a transversal. Bisect two interior angles on the same side of the transversal.

Develop and write a conjecture about the intersection of the angle bisectors.

## Notes

### Materials:

Patty paper and one straightedge per student or geometry software

### Geometry TEKS Focus:

(G.9) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student is expected to:

(A) formulate and test conjectures about the properties of parallel and perpendicular lines based on explorations and concrete models;

### Additional Geometry TEKS:

(G.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student is expected to:

(A) use constructions to explore attributes of geometric figures and to make conjectures about geometric relationships; and

## Teacher Notes

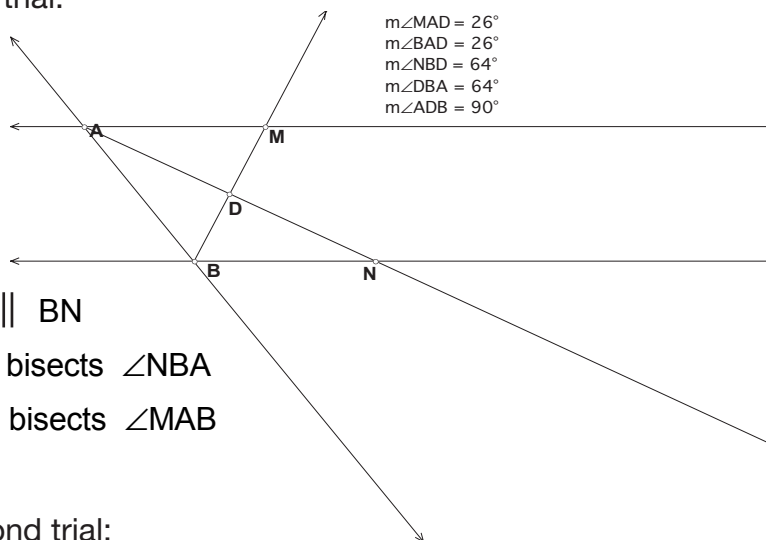
### Scaffolding Questions:

- What does the given information imply about the angles in the figure?
- Which special properties of parallel lines can you use to describe the relationships between or among the angles?

### Sample Solutions:

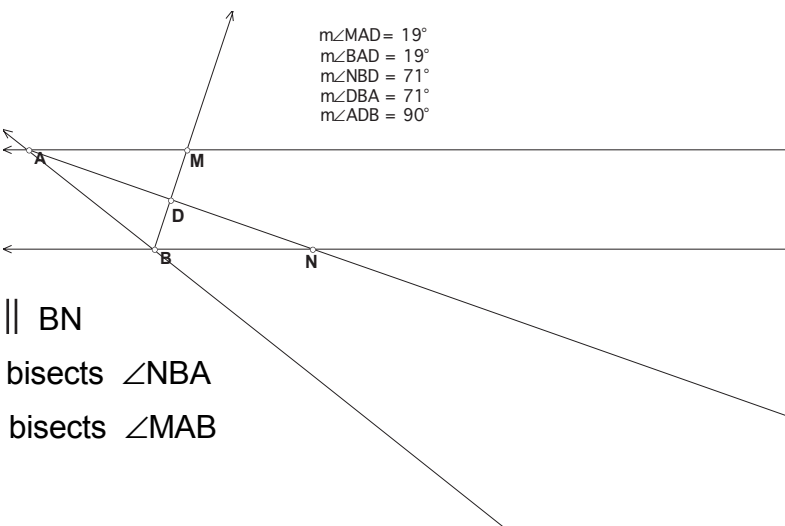
Construct the figure using geometry software. Measure the resulting angles.

First trial:



- AM  $\parallel$  BN
- BD bisects  $\angle NBA$
- AD bisects  $\angle MAB$

Second trial:



- AM  $\parallel$  BN
- BD bisects  $\angle NBA$
- AD bisects  $\angle MAB$

Conjecture: The two angle bisectors are perpendicular.

#### Extension Questions:

- Prove your conjecture using an axiomatic approach.

$$m\angle BAD = \frac{1}{2}m\angle BAM \text{ because AD bisects } \angle BAM .$$

$$m\angle MBA = \frac{1}{2}m\angle ABN \text{ because BD bisects } \angle ABN .$$

The two interior angles on the same side of the transversal of two parallel lines are supplementary.

$$\text{Thus, } m\angle ABN + m\angle BAM = 180^\circ$$

$$\frac{1}{2}m\angle ABN + \frac{1}{2}m\angle BAM = 90^\circ$$

By substitution:

$$m\angle MBA + m\angle BAD = 90^\circ \text{ (Equation 1)}$$

The sum of the angles of a triangle is  $180^\circ$ , so in  $\triangle BDA$

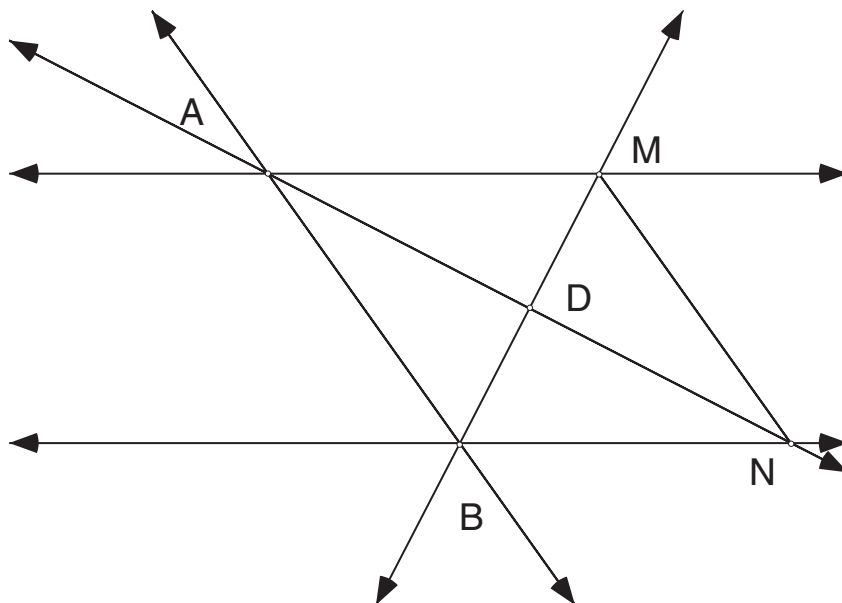
$$m\angle MBA + m\angle BAD + m\angle ADB = 180^\circ . \text{ (Equation 2)}$$

Subtract Equation 1 from Equation 2.

$$m\angle ADB = 90^\circ$$

Therefore  $\angle ADB$  is a right angle and the lines are perpendicular.

- Connect the two points that are the intersection of the angle bisectors and the parallel lines. Describe the resulting triangles and the quadrilateral AMNB.



#### Connections to TAKS:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

The four triangles are congruent, and  $AMNB$  is a rhombus. The triangles  $AMD$  and  $ABD$  are congruent right triangles with common leg  $\overline{AD}$ , congruent angles ( $\angle DAB \cong \angle DAM$ ), and congruent sides ( $\overline{AM} = \overline{AB}$ ).

Also, the triangles  $NBD$  and  $ABD$  are congruent right triangles with common leg  $\overline{BD}$ , congruent angles ( $\angle DBA \cong \angle DBN$ ), and congruent sides ( $\overline{BN} = \overline{AB}$ ).

Therefore,  $AM = AB = BN$ .

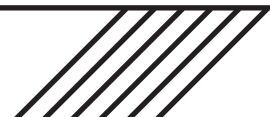
Because  $AM$  is parallel to  $BN$  and  $AM = BN$ , the figure is a parallelogram.

$AB = MN$  because they are opposite sides of the parallelogram.

$MN = AM = AB = BN$

The figure is a rhombus.

Another way to conclude that it is a rhombus is to realize that it is a parallelogram with perpendicular diagonals.



## Student Work Sample

This student's work shows the use of a compass to construct the parallel lines and bisect the angles.

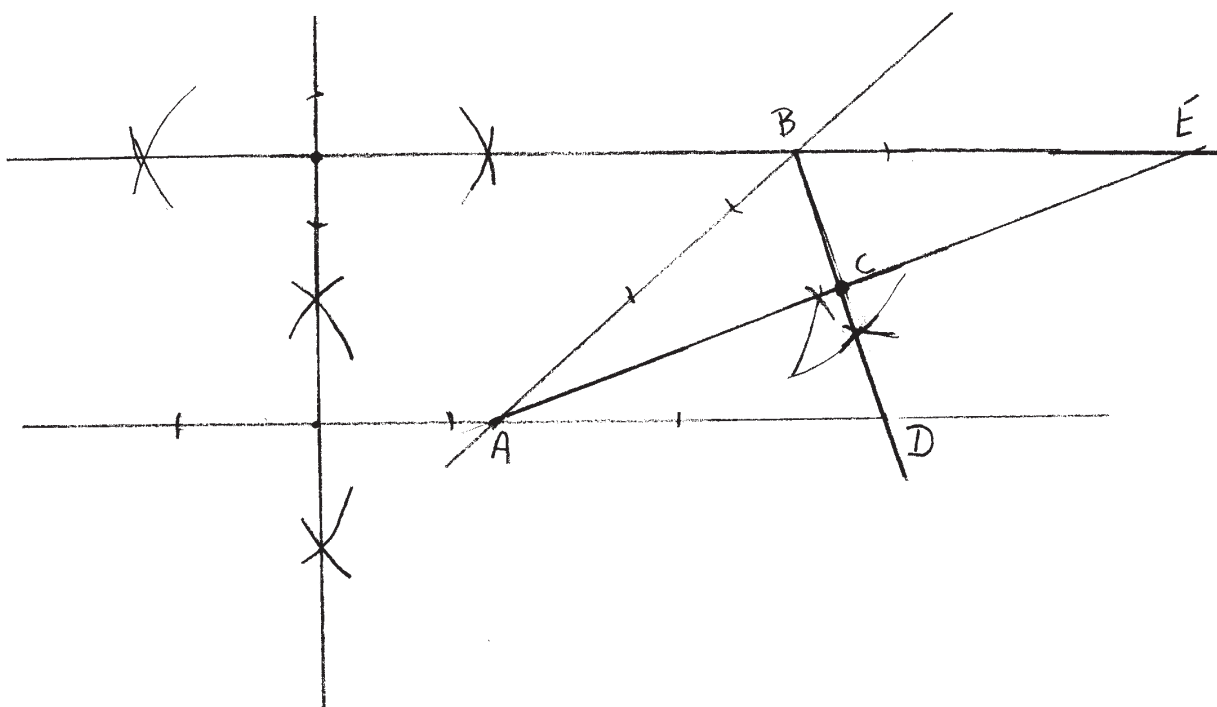
His work exemplifies many of the criteria on the solution guide, especially the following:

- Identifies the important elements of the problem

*The student understands what the problem requires. He understands the lines referenced in the problem—the angle bisectors—and writes his conjecture about these bisectors.*

- States a clear and accurate solution using correct units.

*This problem requires a construction and a conjecture. The student's construction is accurate, and his conjecture is clearly stated. Note the problem does not require that the student justify his conjecture, but he writes an explanation of how he came to his conjecture. The solution does not require units, but the student used the correct units in his justification.*



$$\begin{aligned} \angle BAD + \angle ABE &= 180^\circ \\ \frac{1}{2} \angle BAD + \frac{1}{2} \angle ABE &= 90^\circ \\ - \quad \frac{1}{2} \angle BAD + \frac{1}{2} \angle ABE + \angle C &= 180^\circ \\ - \angle C &= -90^\circ \\ \angle C &= 90^\circ \end{aligned}$$

When a transversal intersects a pair of parallel lines, and two of the interior angles on the same side are bisected, the bisectors are perpendicular.

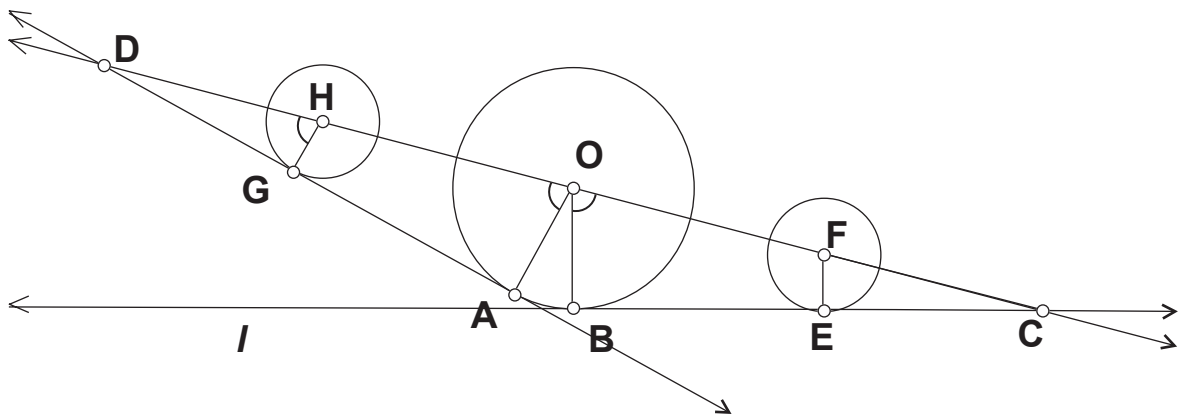




## Circles and Tangents

Maren constructed the figure below by going through the following process:

- Draw a line,  $l$ .
- Construct a circle with center  $O$  tangent to the line at point  $B$ .
- Construct a smaller circle with center  $F$  that is tangent to line  $l$  at point  $E$ . Draw  $\overleftrightarrow{OF}$ . The intersection of line  $l$  and  $\overleftrightarrow{OF}$  is point  $C$ .
- Construct another circle on the other side of line  $l$ , such that the new circle is congruent to circle  $F$  and has center  $H$  on  $\overleftrightarrow{OF}$ , such that  $HO = FO$ .
- Construct two angles congruent to angle  $COB$ , one with center  $O$  and another with center  $H$  as shown in the diagram.
- The intersection point with circle  $O$  and the angle ray is point  $A$  and the intersection point of the angle ray and circle  $H$  is  $G$ .
- Draw the line  $\overleftrightarrow{GA}$ .



Develop and write a conjecture about the relationships among the angles and the triangles in the figure and between the circles and the line  $\overleftrightarrow{GA}$ .

## Teacher Notes

**Materials:**

One straightedge and compass per student or geometry software

**Geometry TEKS Focus:**

(G.9) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student is expected to:

(C) formulate and test conjectures about the properties and attributes of circles and the lines that intersect them based on explorations and concrete models; and

**Additional Geometry TEKS:**

(G.11) **Similarity and the geometry of shape.** The student applies the concepts of similarity to justify properties of figures and solve problems.

The student is expected to:

(A) use and extend similarity properties and transformations to explore and justify conjectures about geometric figures;

**Connections to TAKS:**

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

**Scaffolding Questions:**

If the student uses constructions to analyze the problem, the following questions may be asked.

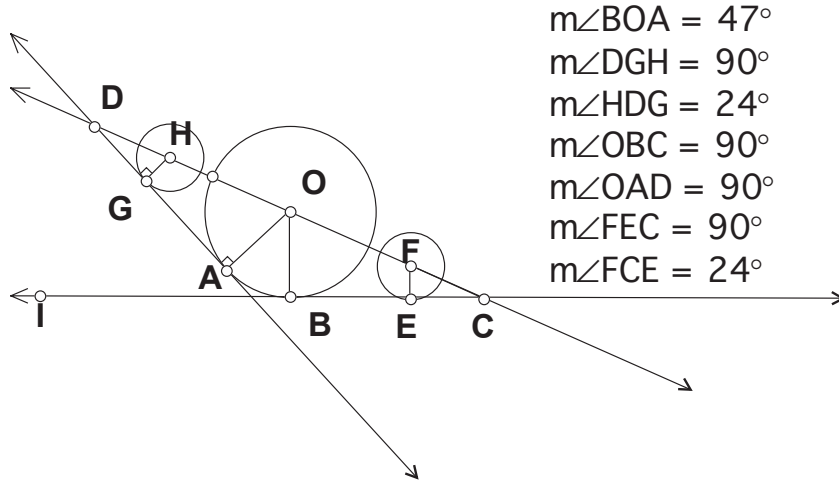
- What is the definition of the tangent to a circle?
- Given a line how can you construct a circle tangent to the line?
- What is the relationship between  $\overline{GH}$  and  $\overline{OA}$ ?
- Which segments in the figure must be congruent?
- Describe the relationship between  $\angle DHG$  and  $\angle CFE$ .
- Explain what you know about the relationship between  $\triangle DGH$  and  $\triangle CEF$ .
- What is special about  $\angle DGH$ ?
- What does it tell you about the figure?

If the student does not have access to geometry software, the problem may be approached analytically.

- What does the given information imply about the triangles in the figure?
- Which special properties of right triangles can you use to describe the relationships between or among the segments in the figure?
- What is the relationship between a central angle and its intercepted arc?

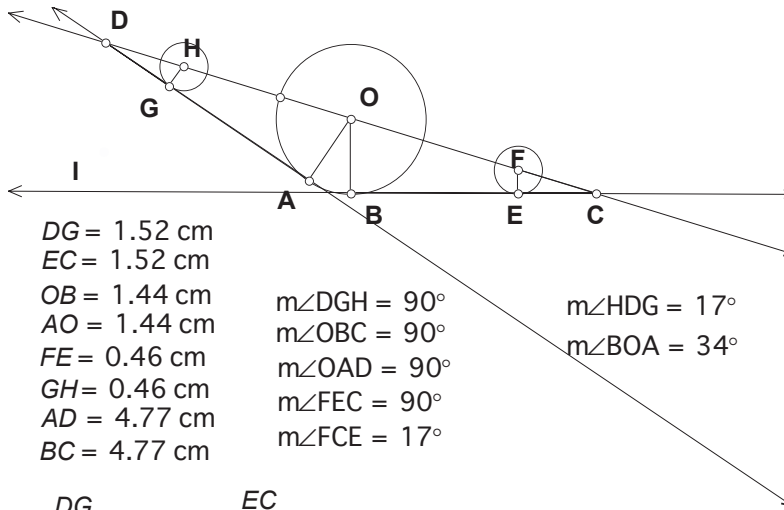
## Sample Solutions:

Construct the figure using geometry software.



- $m\angle BOA = 47^\circ$
- $m\angle DGH = 90^\circ$
- $m\angle HDG = 24^\circ$
- $m\angle OBC = 90^\circ$
- $m\angle OAD = 90^\circ$
- $m\angle FEC = 90^\circ$
- $m\angle FCE = 24^\circ$

Try a second example. Measure the angles and the ratios of the corresponding sides of triangles.



- $DG = 1.52 \text{ cm}$
- $EC = 1.52 \text{ cm}$
- $OB = 1.44 \text{ cm}$
- $AO = 1.44 \text{ cm}$
- $FE = 0.46 \text{ cm}$
- $GH = 0.46 \text{ cm}$
- $AD = 4.77 \text{ cm}$
- $BC = 4.77 \text{ cm}$

- $m\angle DGH = 90^\circ$
- $m\angle OBC = 90^\circ$
- $m\angle OAD = 90^\circ$
- $m\angle FEC = 90^\circ$
- $m\angle FCE = 17^\circ$

- $m\angle HDG = 17^\circ$
- $m\angle BOA = 34^\circ$

$$\frac{DG}{AD} = 0.32 \quad \frac{EC}{BC} = 0.32$$

$$\frac{FE}{OB} = 0.32 \quad \frac{GH}{AO} = 0.32$$

Possible Conjectures:

$\overleftrightarrow{GA}$  is tangent to circles O and H.

The angle BOA is equal to twice the measure of the angle FCE or angle GDH.

$$\triangle CFE \cong \triangle DHG$$

$$\triangle CBO \cong \triangle DAO$$

$$\triangle CBO \approx \triangle CEF$$

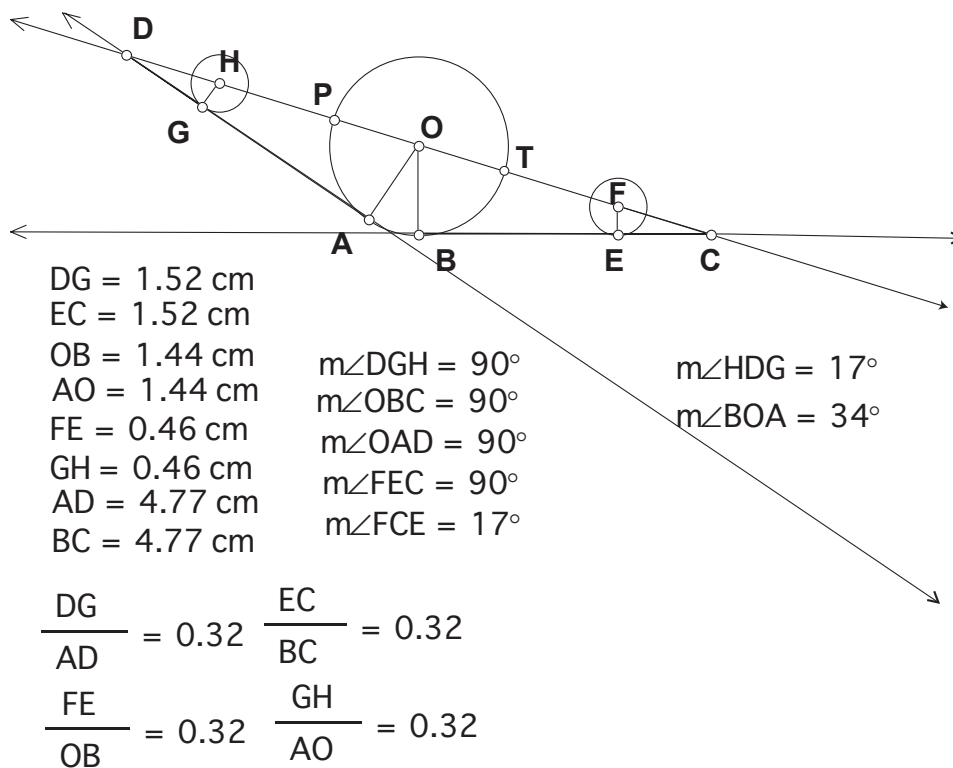
$$\triangle DAO \approx \triangle DGH$$

**Extension Questions:**

- How would your conjecture change if the circles were tangent or intersecting?

The relationships are the same no matter what the position of the circles.

- Prove that the measure of  $\angle AOB$  is twice the measure of  $\angle ECF$ .



The proof will depend upon the theorems that have previously been developed in the geometry classroom. One possible proof follows:

$$m\angle POA \cong m\angle BOT$$

Given (constructed to be congruent)

$$m\angle TOP = 180^\circ$$

PT is a diameter of the circle

$$m\angle POA + m\angle AOB + m\angle BOT = m\angle TOP$$

An angle is the sum of its parts

$$m\angle BOT + m\angle AOB + m\angle BOT = 180^\circ$$

Substitution

$\overline{BC}$  is tangent to circle O

Given

$\overline{OB} \perp BC$

Definition of tangent

$$m\angle OBC = 90^\circ$$

Definition of perpendicular

$$m\angle BOT + m\angle ECF = 90^\circ$$

is 90

Sum of the acute angles of a triangle  
degrees.

$$2(m\angle BOT + m\angle ECF) = 180^\circ$$

Multiplication

$$m\angle BOT + m\angle AOB + m\angle BOT =$$

Substitution

$$2(m\angle BOT) + 2(m\angle ECF)$$

$$m\angle AOB = 2(m\angle ECF)$$

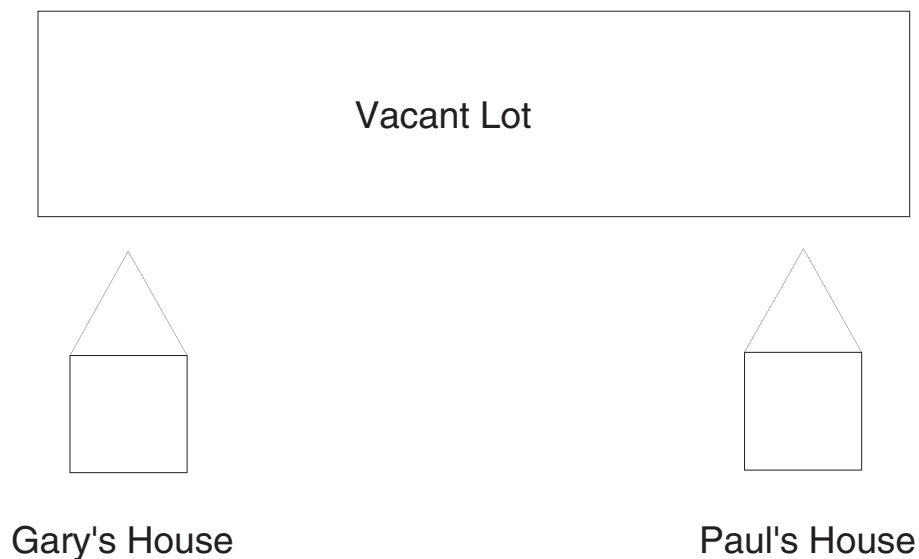
Subtraction



## The Clubhouse

Gary and Paul want to build a clubhouse on the vacant lot near their houses. To be fair, they decide it has to be the same distance from both of their houses. Describe all the possible locations of the clubhouse so that it is the same distance from both of their houses and in the vacant lot. Justify your answer.

Include a discussion of the properties of the lines and figures you have created. Can you generalize your conjecture?



## Teacher Notes

### One teacher says . . .

“It’s hard for some teachers to let students struggle before answering questions for the students. We as teachers need to allow them to struggle and ask their teammates before answering questions. This allows them to work on teamwork skills needed for the work place in their future.”

### Materials:

Tracing paper and one straightedge per student

### Geometry TEKS Focus:

(G.9) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student is expected to:

(A) formulate and test conjectures about the properties of parallel and perpendicular lines based on explorations and concrete models;

### Additional Geometry TEKS:

(G.10) **Congruence and the geometry of size.** The student applies the concept of congruence to justify properties of figures and solve problems.

The student is expected to:

(A) use congruence transformations to make conjectures and justify properties of geometric figures including figures represented on a coordinate plane; and

### Scaffolding Questions:

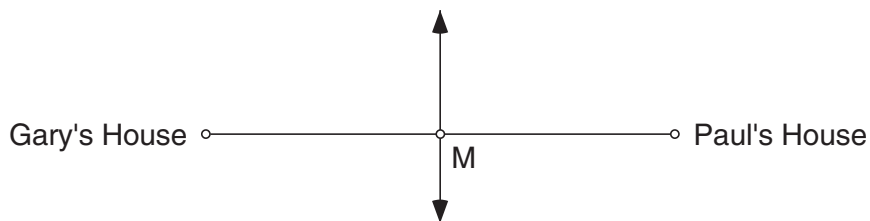
- What are some strategies that you could use to solve this problem?
- Is there more than one location in the vacant lot where they could place the clubhouse?

### Sample Solutions:

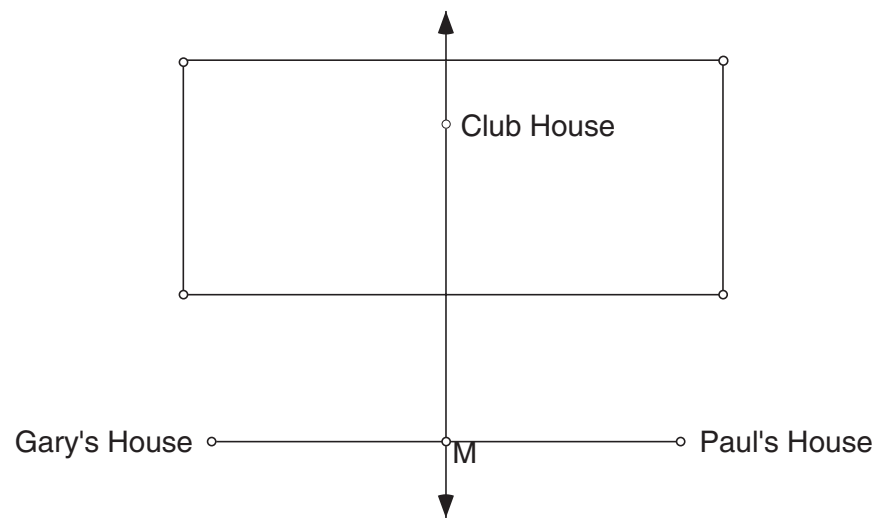
Draw a segment connecting the two houses.

Gary's House ○ ————— ○ Paul's House

Construct the perpendicular bisector of the connecting segment. Label the intersection of the connecting segment and its perpendicular bisector point M.

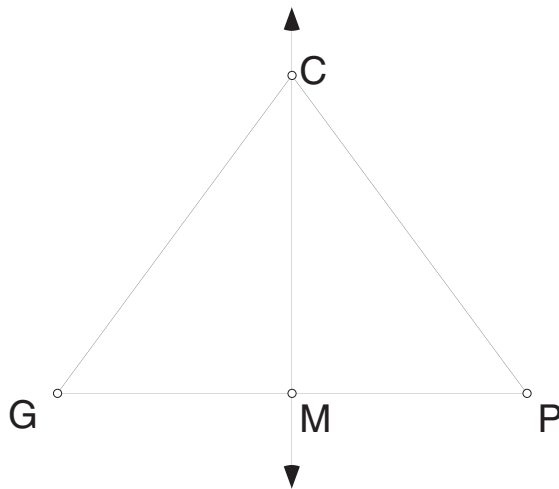


Place the clubhouse anywhere on this perpendicular bisector and within the lot.





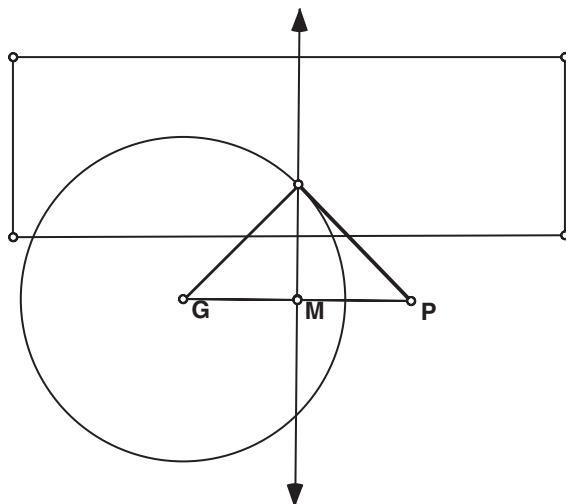
Draw segments connecting their houses to the clubhouse.



Because  $\overleftrightarrow{CM}$  is the perpendicular bisector of  $\overline{GP}$ ,  $\overline{GM} \cong \overline{MP}$ , and  $\triangle GMC$  and  $\triangle PMC$  are right triangles with common leg  $\overline{CM}$ , the two right triangles are congruent. Therefore,  $\overline{GC} \cong \overline{PC}$ . This could also be demonstrated by reflecting the segment connecting Gary's house to the clubhouse onto the segment connecting Paul's house to the clubhouse by folding along the perpendicular bisector. The segments are congruent so any point on the perpendicular bisector is equidistant from the endpoints of the segment it bisects.

**Extension Questions:**

- Suppose you have selected the position for the clubhouse. Are there other points in the vacant lot that are the same distance away from Gary's house as the distance between Gary's house and the clubhouse?



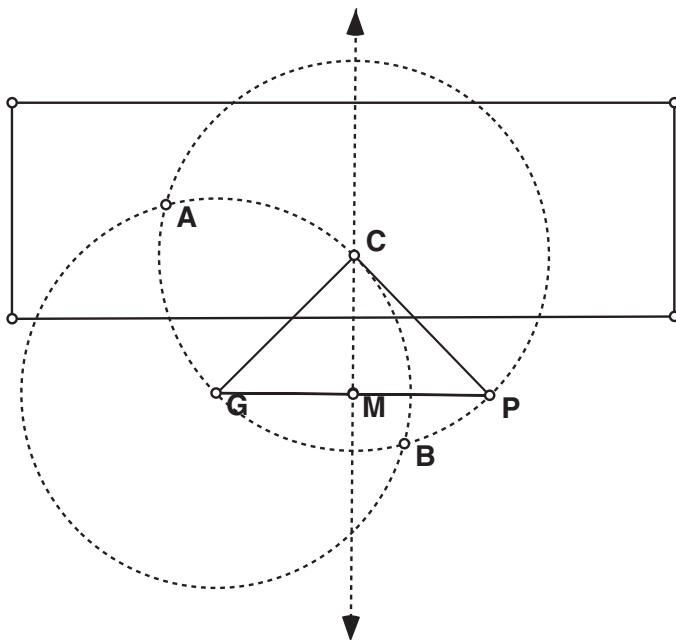
**Connections to TAKS:**

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Any point on the circle with center  $G$  and radius  $\overline{GC}$  that is in the vacant will be the same distance from Gary's house as the clubhouse is from Gary's house.

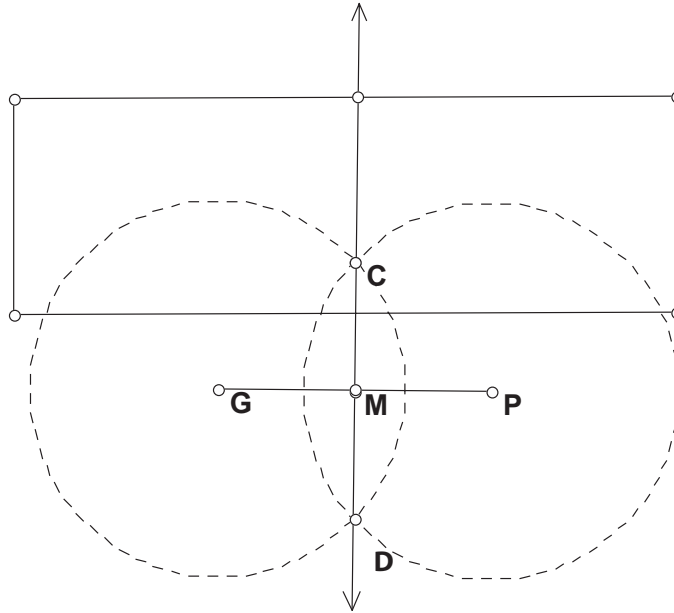
- Describe the location of the points such that the distance from the points to the clubhouse is the same as the distance from Gary's house to the clubhouse.

Using  $C$  as a center point and  $\overline{GC}$  as the radius construct a circle. The intersections of this circle and circle  $G$  are the points  $A$  and  $B$ , which are the same distance from  $G$  and from  $C$ . One of these points could be in the vacant lot.

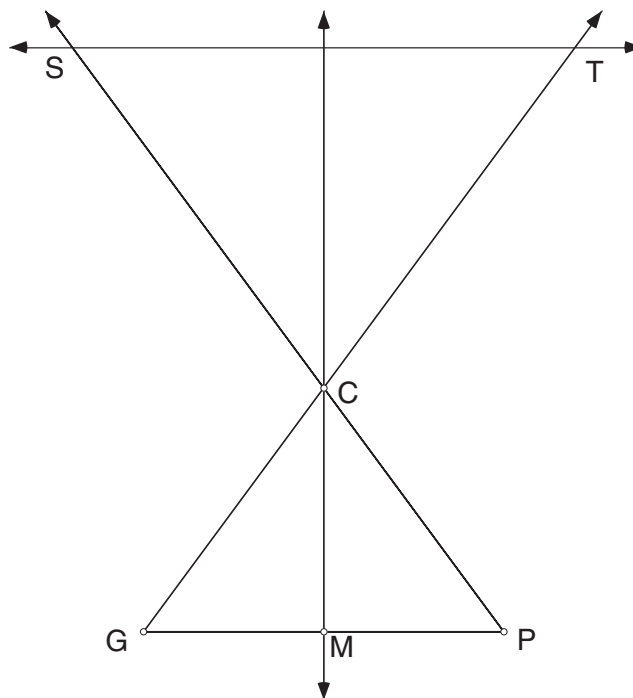


- Is there a point that is the same distance away from  $G$  and  $P$  as the clubhouse?

There is another point,  $D$ , which is on the intersection of the two circles with centers at  $G$  and  $P$ , but it would be in the opposite direction as  $C$  and would not be in the vacant lot.



- Suppose that a street runs parallel to the segment connecting Gary and Paul's houses, and it is on the other side of the vacant lot. Gary and Paul both walk from their own house through the clubhouse to the street as shown in the diagram. Describe anything you can about the resulting figures.



Possible answers:

If  $\overrightarrow{ST} \parallel \overrightarrow{GP}$ , then the alternate interior angles are congruent.

$$\angle S \cong \angle P \text{ and } \angle T \cong \angle G$$

Vertical angles are congruent,  $\angle SCT \cong \angle GCP$ .

Two similar triangles are formed because three angles of one triangle are congruent to three angles of the other triangle.

$$\triangle SCT \approx \triangle PCG$$

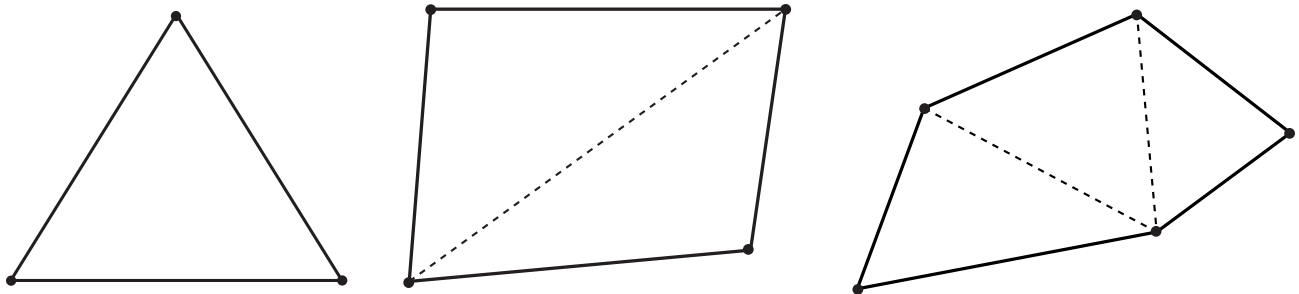
Because  $CG = CP$ , then triangle  $GCP$  is isosceles.

Since  $\triangle SCT \approx \triangle PCG$ , triangle  $SCT$  is also isosceles and  $CS = CT$ .

The distance Gary would walk would be equal to the distance Paul would walk.

## Diagonals and Polygons

Write a conjecture about the relationship between the number of non-intersecting diagonals and the sum of the interior angles in a convex polygon. Justify your conjecture. Represent this function using symbols and a graph.



Polygon Name	Number of Sides	Number of Diagonals from One Vertex	Process to Find the Sum of the Interior Angles	Sum of the Interior Angles (in degrees)
		0		
		1		
		2		
		3		
		4		
		10		
		$n$		



## Notes

### Materials:

One straightedge, protractor, and graphing calculator per student

### Geometry TEKS Focus:

(G.5) **Geometric patterns.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to:

(B) use numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles;

(G.9) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student is expected to:

(B) formulate and test conjectures about the properties and attributes of polygons and their component parts based on explorations and concrete models;

### Additional Geometry TEKS:

(G.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student is expected to:

(B) make conjectures about angles, lines, polygons, circles, and three-dimensional figures and determine the validity of the conjectures, choosing from a variety of approaches such as

## Teacher Notes

### Scaffolding Questions:

- How many triangles do the diagonals form?
- What do you know about the sum of the interior angles of a triangle?
- What do you already know about the sum of the interior angles of a polygon?
- How do the number of sides, the number of triangles, and the number of diagonals from one vertex relate to the sum of the interior angles?

### Sample Solutions:

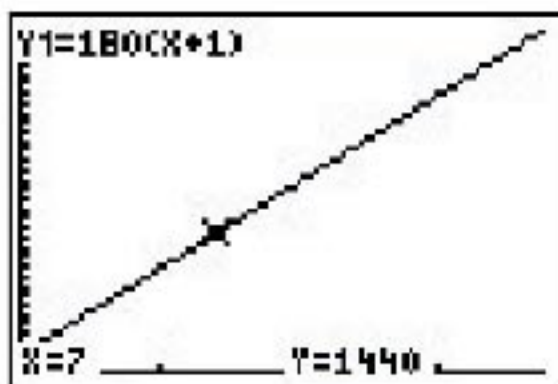
Draw the convex polygons; draw the diagonals. The only diagonals that satisfy the requirement to be non-intersecting are the ones drawn from one vertex. If another diagonal is drawn from a different vertex, it will intersect one of the diagonals from the first vertex. Count the number of triangles. There seems to be one more triangle than there are diagonals. To find the sum of their interior angles multiply the number of triangles by 180 degrees.

Complete the table.

Polygon Name	Number of Sides	Number of Diagonals from One Vertex	Process to Find the Sum of the Interior Angles	Sum of the Interior Angles (in degrees)
Triangle	3	0	180 or $180(0+1)$	180
Quadrilateral	4	1	$180(2)$ or $180(1+1)$	360
Pentagon	5	2	$180(3)$ or $180(2+1)$	540
Hexagon	6	3	$180(4)$ or $180(3+1)$	720
Heptagon	7	4	$180(5)$ or $180(4+1)$	900
Tridecagon		10	$180(10+1)$ or $180(11)$	1980
n-gon		$n$	$180(n+1)$	

The number of triangles is 1 more than the number of diagonals. The sum of the interior angles of a convex polygon is 180 times 1 more than the number of non-intersecting diagonals. If  $s$  represents the sum of the angles of the polygon, and  $n$  represents the number of non-intersecting diagonals, then

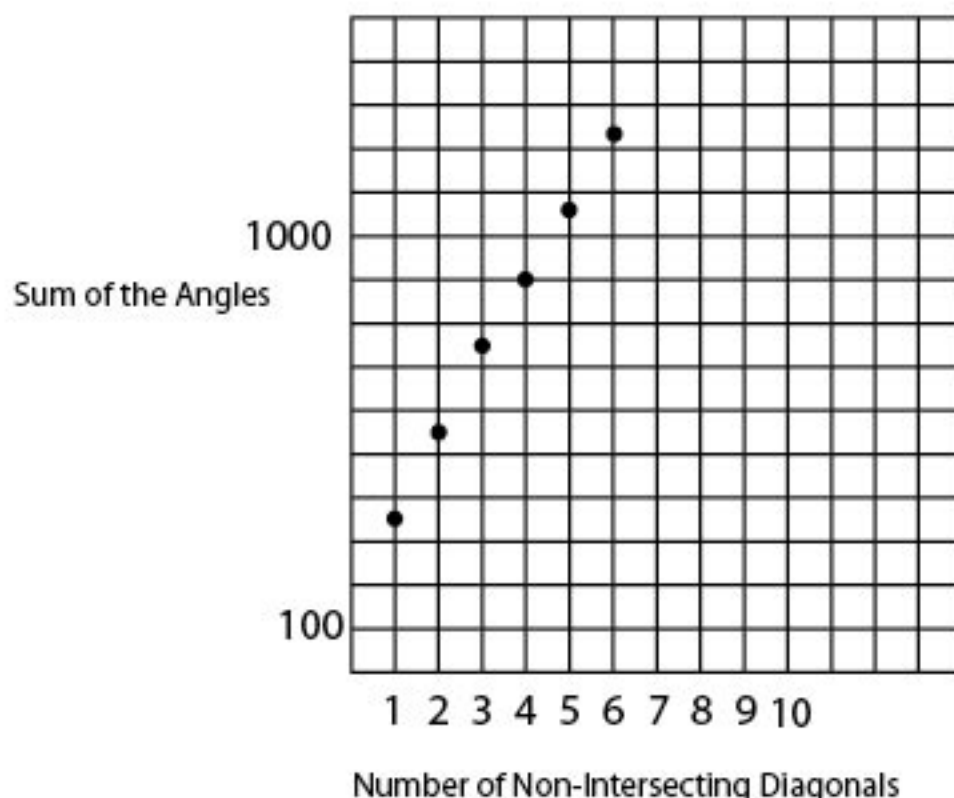
$$s = 180(n + 1).$$



#### Extension Questions:

- What are the restrictions on the domain of the function?

The domain values must be the counting numbers. The graph should be a set of points.



coordinate, transformational, or axiomatic.

(G.3) **Geometric structure.** The student applies logical reasoning to justify and prove mathematical statements.

The student is expected to:

(D) use inductive reasoning to formulate a conjecture; and

(G.5) **Geometric patterns.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to:

(A) use numeric and geometric patterns to develop algebraic expressions representing geometric properties;

#### Connections to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

- > How does the number of diagonals from one vertex relate to the number of sides? ¶
  - > There are 3 fewer diagonals than sides. If  $n$  is the number of diagonals, the number of sides is expressed as  $n + 3$ . ¶
- > Will there be a polygon such that the sum of the angles of the polygon is 2000 degrees? Explain why or why not. ¶
  - > The table shows that the polygon with 10 diagonals has a sum of angles of 1980 degrees and the polygon with 11 diagonals has a sum of angles of 2160 degrees. ¶
  - > Since the number of diagonals must be a whole number, there is no polygon with an angle sum of 2000 degrees. ¶

X	Y <sub>1</sub>	
8	1620	
9	1800	
10	1980	
11	2160	
12	2340	
13	2520	
14	2700	
X=10		

- > Use your table or graph to determine the sum of the angles of a polygon with 20 diagonals. ¶
  - > The sum of the angles of a polygon with 20 diagonals is 3780 degrees. ¶

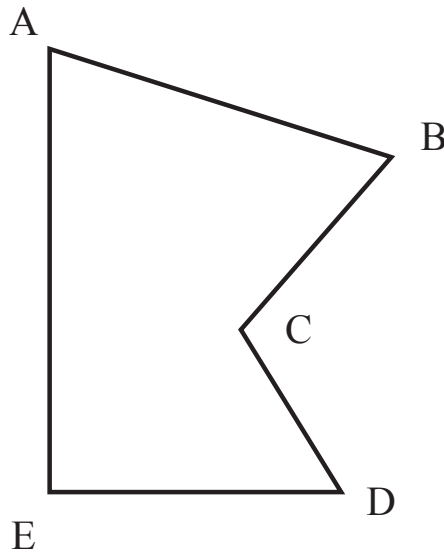
X	Y <sub>1</sub>	
15	2880	
16	3060	
17	3240	
18	3420	
19	3600	
20	3780	
21	3960	
X=20		

- > Does the relationship hold true if the polygons are concave? ¶

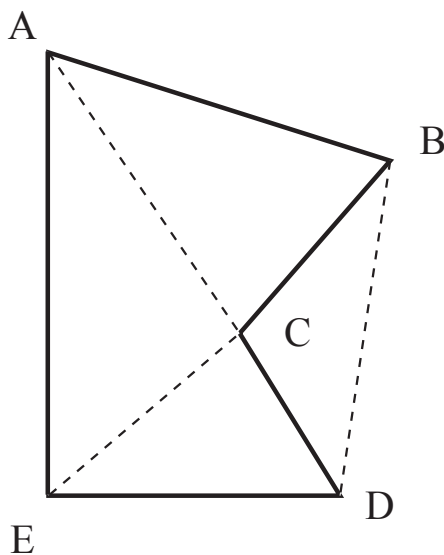


The relationship does not hold true for concave polygons. See the example below.

This is a concave polygon. A diagonal is a segment that has endpoints that are non-adjacent vertices of the polygon.



Draw non-intersecting diagonals



If the formula is applied using the number 3, the sum of the interior angles is  $(3+1)180 = 4(180)$  or 720 degrees. The sum of the interior angles of this polygon is not 720 degrees. To determine the sum, note that the quadrilateral is composed of three triangles,  $\triangle ACE$ ,  $\triangle ACB$ , and  $\triangle ECD$ . The sum of the angles of the 3 triangles is  $3(180)$  or 540 degrees. The formula does not work for this concave polygon.

## Student Work Sample

This student completed the given table and then wrote the explanation and constructed the graph on the next page. This student's work exemplifies the following criteria from the solution guide:

- Evaluates reasonableness or significance of the solution in the context of the problem.

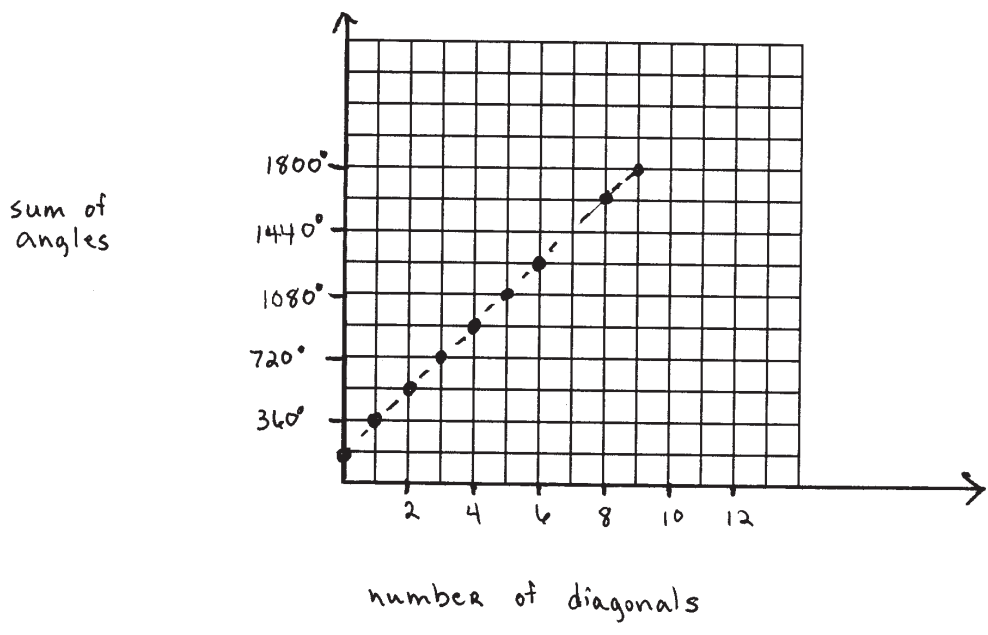
*The graph is an accurate representation of the situation. The student plots points on the graph and shows a dashed line to indicate that the whole line does not represent the situation. (The number of diagonals must be a whole number.)*

- Uses appropriate terminology and notation.

*In his conjecture, the student identifies the variables, describes the relationships using an accurate verbal description of the relationship between the variables.*

### Diagonals and Polygons

A polygon with  $n$  diagonals from each vertex can be split into  $n + 1$  triangles. Since triangles have an interior angle sum of  $180^\circ$ , and the triangles interior  $\angle$ s make up the larger polygons interior  $\angle$ s, the sum of the interior angles is  $180^\circ$  times the number of triangles formed by the diagonals of one vertex, or (the number of diagonals from 1 vertex + 1) times  $180^\circ$ .





## The Most Juice

You are in charge of buying containers for a new juice product called Super-Size Juice. Your company wants a container that provides the greatest volume for the given parameters.

- A six-inch straw must touch every point on the base with at least one inch remaining outside of the container.
- The base of the container must either be a square, 4 inches by 4 inches, or a circle that would be inscribed in that square.
- The hole that the straw is inserted into in the top of the container must be exactly one inch from the side of the cylinder or from both sides of the prism.

You have two containers to choose from—a rectangular prism or a right cylinder. Which one would you choose? Justify your answer.

## Teacher Notes

### Materials:

One straightedge and one graphing calculator per student

### Geometry TEKS Focus:

(G.8) **Congruence and the geometry of size.** The student uses tools to determine measurements of geometric figures and extends measurement concepts to find area, perimeter, and volume in problem situations.

The student is expected to:

(D) find surface areas and volumes of prisms, pyramids, spheres, cones, cylinders, and composites of these figures in problem situations.

(G.9) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student is expected to:

(D) analyze the characteristics of polyhedra and other three-dimensional figures and their component parts based on explorations and concrete models.

### Additional Geometry TEKS:

(G.5) **Geometric patterns.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student is expected to:

(D) identify and apply patterns from right triangles to solve meaningful problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

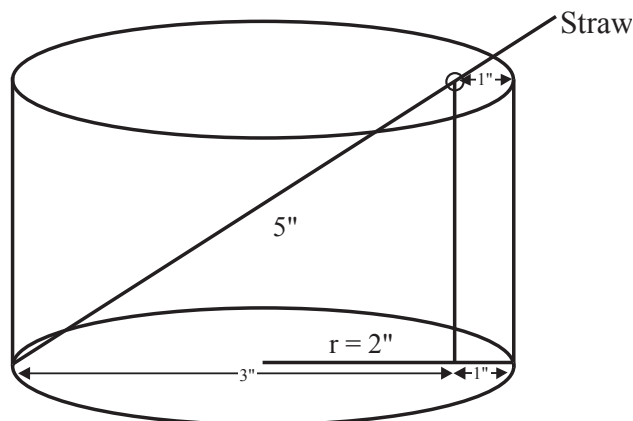
### Scaffolding Questions:

- Can you place the hole for the straw in more than one place?
- Does one inch from the side mean one inch from every side?
- What would pictures of the various containers and hole positions look like?

### Sample Solutions:

For the cylinder:

If the base of the figure is a circle inscribed in a square of sides 4 inches, then the diameter of the inscribed circle is 4 inches. The cylinder has a radius of 2 inches.



If the hole is 1 inch from the edge, a segment from the hole through the center to the other side of the cylinder is  $4 - 1$  or 3 inches. One inch of the straw is outside of the can, so 5 inches are inside the can. When meeting the parameters for the straw, a right triangle with a leg of 3 inches and a hypotenuse of 5 inches is formed. This is a Pythagorean triple, so the other leg must be 4 inches. This makes the minimum height for the cylinder 4 inches.

The volume is:

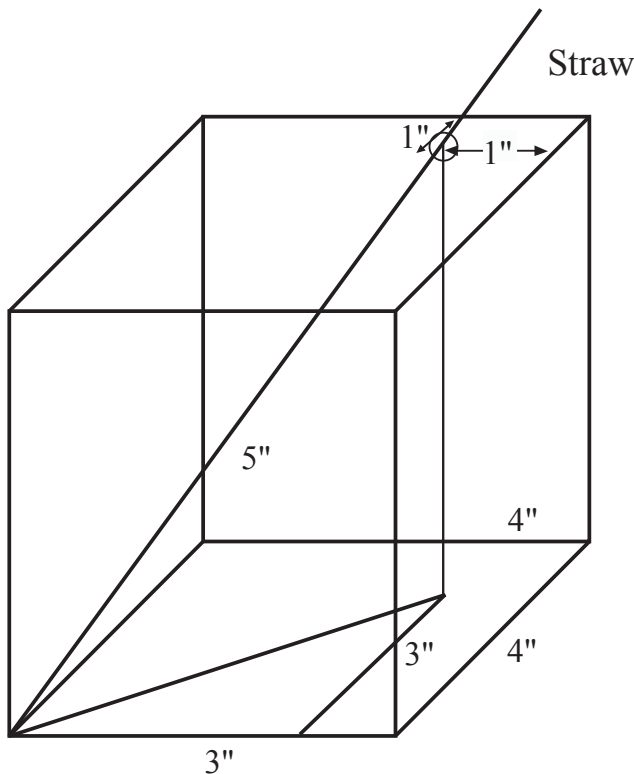
$$V = Bh$$

$$V = \pi r^2 h$$

$$V \approx 3.14 \times 2^2 \times 4$$

$$V \approx 50.24 \text{ in}^3$$

For the rectangular prism with a square base:



The triangle is a right triangle with legs of measure 3 inches. The hypotenuse is  $3\sqrt{2}$ . Use the Pythagorean Theorem to determine the height of the box.

$$(3\sqrt{2})^2 + h^2 = 5^2$$

$$18 + h^2 = 25$$

$$h^2 = 7$$

$$h \approx 2.65 \text{ in}$$

(G.8) **Congruence and the geometry of size.** The student uses tools to determine measurements of geometric figures and extends measurement concepts to find area, perimeter, and volume in problem situations.

The student is expected to:

(C) derive, extend, and use the Pythagorean Theorem; and

**Connections to TAKS:**

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of concepts and uses of measurement and similarity.

**One teacher says . . .**

“The students needed clarification on the straw placement. I asked them about drinking out of a cup with a straw and how the straw reaches inside the cup to the corner.”

Now find the volume of this box.

$$V = Bh$$

$$V = s^2h$$

$$V \approx 4^2 \times 2.65$$

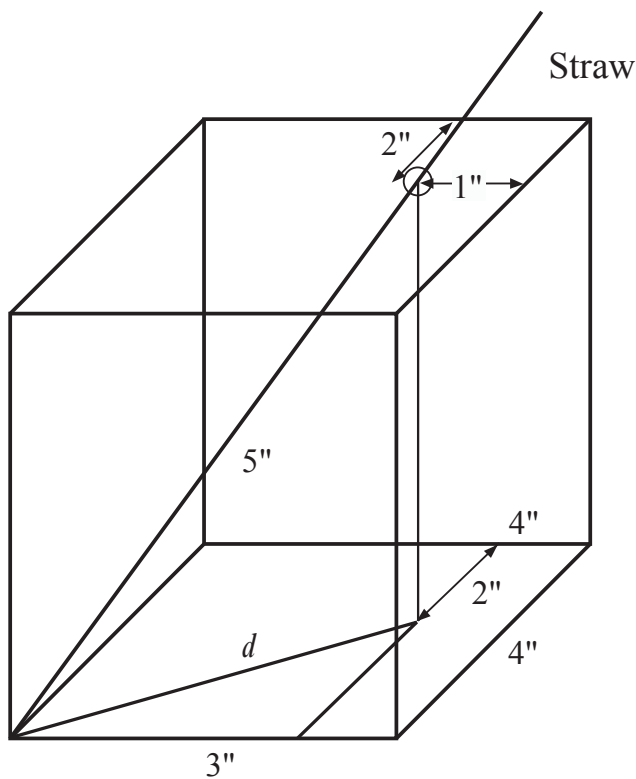
$$V \approx 42.4 \text{ in}^3$$

The volume of the cylinder is approximately  $50.24 \text{ in}^3$ ; the cylinder has the larger volume and is a better choice than the rectangular prism.

**Extension Questions:**

- Suppose that the straw must be 1 inch from one edge and 2 inches from the other (perpendicular) edge. How does the volume of this box compare to the previous two choices?

This situation yields the largest volume and places the hole 1 inch from one side in the center of the box.



Use the Pythagorean Theorem to find the hypotenuse,  $d$ , of the right triangle in the base of the box.



$$\begin{aligned}a^2 + b^2 &= c^2 \\ \sqrt{a^2 + b^2} &= c \\ \sqrt{2^2 + 3^2} &= c \\ \sqrt{13} &= c \\ c &= \sqrt{13} \text{ in}\end{aligned}$$

The base leg of the vertical right triangle is  $d$ . Now use the Pythagorean Theorem again to find the height of the box.

$$\begin{aligned}a^2 + b^2 &= c^2 \\ (\sqrt{13})^2 + b^2 &= 5^2 \\ 13 + b^2 &= 25 \\ b^2 &= 12 \\ b &\approx 3.46 \text{ in}\end{aligned}$$

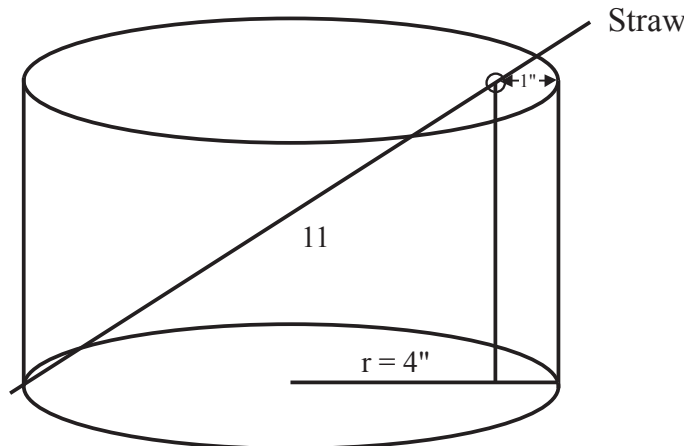
Now find the volume of this box.

$$\begin{aligned}V &= Bh \\ V &= s^2h \\ V &\approx 4^2 \times 3.46 \\ V &\approx 55.36 \text{ in}^3\end{aligned}$$

This box is the best choice because it has the greatest volume.

- Suppose that the length of the straw and the diameter of the base of the cylinder are doubled. How will the volume be affected?

Usually if the dimensions of a solid are doubled, the volume is affected by a factor of 2 cubed. However, there are other factors to consider. The diameter of the base is 8 inches. The cylinder has a radius of 4 inches.



If the hole is 1 inch from the edge, a segment from the hole through the center to the other side of the cylinder is  $8 - 1$  or 7 inches. The part of the straw that is inside the can is  $12 - 1$  or 11 inches.

The height is found using the Pythagorean Theorem.

$$11^2 = 7^2 + h^2$$

$$h^2 = 121 - 49 = 72$$

$$h = \sqrt{72} \approx 8.48$$

The volume is:

$$V = Bh$$

$$V = \pi r^2 h$$

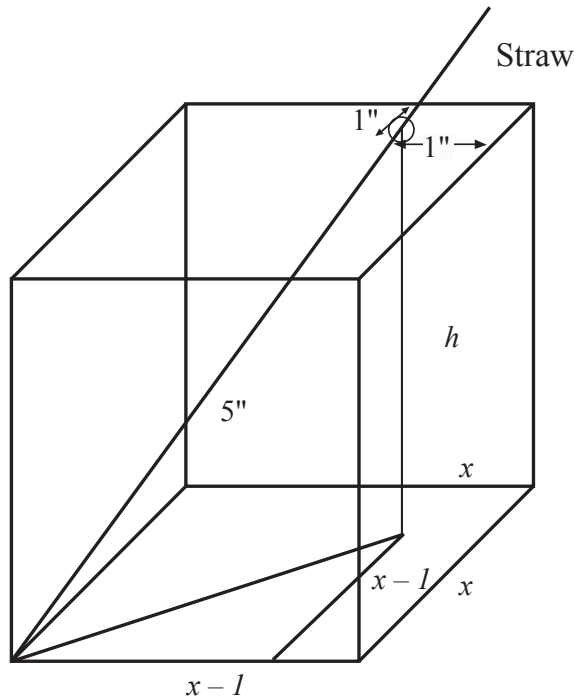
$$V \approx 3.14 \times 4^2 \times 8.48$$

$$V \approx 426.04 \text{ in}^3$$

This number is more than 8 times 50.24 cubic inches. Not all of the dimensions were changed by a multiple of 2, so the theory of changing the volume by a factor of 2 cubed does not apply. Since the height of the new cylinder is 2.12 times the height of the original cylinder, the volume of the new cylinder is  $2(2)(2.12)$  times the volume of the original cylinder.

- Suppose that the base of the rectangular prism could be any square that had dimension 4 inches or less. If the straw hole is one inch from both sides of the base, what can you tell about the variables and the volume?

Let the side of the square base be  $x$  inches. The length of one side of the right triangle shown on the base is represented by  $(x - 1)$  inches.



The hypotenuse of the right triangle is  $(x - 1)\sqrt{2}$ .

The relationship between the side,  $x$ , and the height,  $h$ , is  $5^2 = [(x - 1)\sqrt{2}]^2 + h^2$ .

The equation may be solved for  $h$ .

$$h^2 = 5^2 - [(x - 1)\sqrt{2}]^2$$

$$h = \pm \sqrt{5^2 - [(x - 1)\sqrt{2}]^2}$$

$$h = \pm \sqrt{25 - 2(x - 1)^2}$$

Because the height cannot be negative,  $h = +\sqrt{25 - 2(x - 1)^2}$ .

The volume of the box is

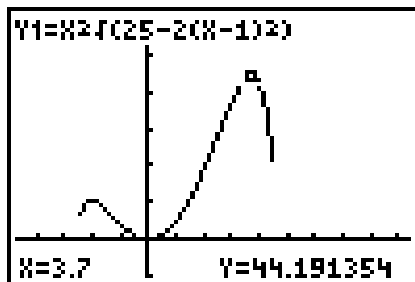
$$V = x^2h$$

$$V = x^2 \sqrt{25 - 2(x - 1)^2}$$

It is possible to graph this function to realize that the maximum volume occurs when  $x$  is about 3.7 inches. Under the given restrictions that  $x$  be less than or equal to 4, a base of about 3.7 inches gives the greatest volume.

```

WINDOW
Xmin=-4.7
Xmax=9.4
Xscl=1
Ymin=-10
Ymax=60
Yscl=10
Xres=1
    
```



X	Y <sub>1</sub>
3.5	43.31
3.6	43.911
3.7	44.191
3.8	44.083
3.9	43.502
4	42.332
4.1	40.414

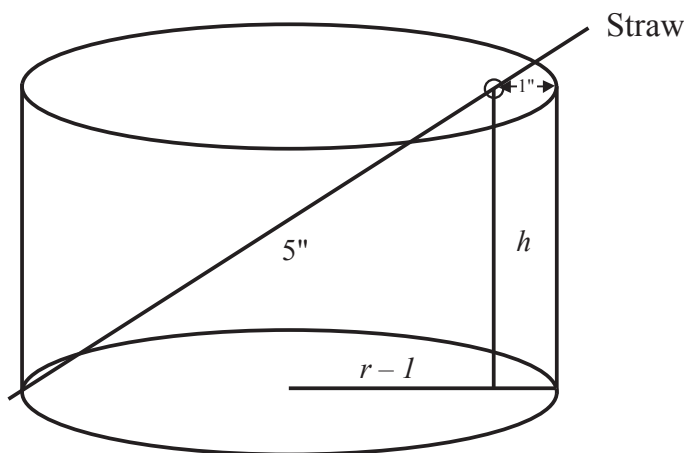
X=3.7

Making the increments on the table smaller would show that the value is between 3.72 and 3.73 inches.

X	Y <sub>1</sub>
3.7	44.191
3.71	44.199
3.72	44.203
3.73	44.203
3.74	44.199
3.75	44.191
3.76	44.178

X=3.72

- If the base may be a circle with a diameter that is less than or equal to 4, what can you conclude about the volume?



Let the radius of the base be represented by  $r$  and let the height of the cylinder be represented by  $h$ .

The right triangle with hypotenuse 5 inches has legs represented by  $2r - 1$  and  $h$ . Apply the Pythagorean Theorem.

$$h^2 + (2r - 1)^2 = 5^2$$

$$h^2 = 5^2 - (2r - 1)^2$$

$$h = \pm \sqrt{5^2 - (2r - 1)^2}$$

$$h = \pm \sqrt{25 - (2r - 1)^2}$$

The height must be positive.  $h = \sqrt{25 - (2r - 1)^2}$

The volume of the cylinder is

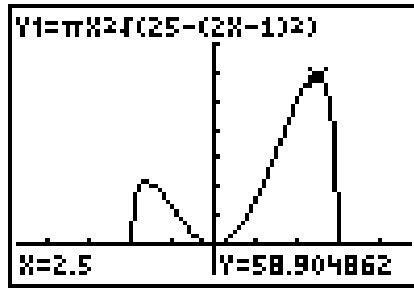
$$V = \pi r^2 h$$

$$V = \pi r^2 \sqrt{25 - (2r - 1)^2}$$

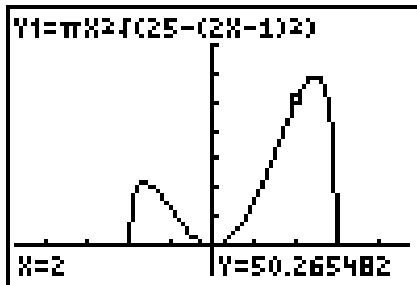
The graph or table of this function illustrates that the maximum volume occurs at a radius of 2.5 inches. However, because of the given restriction that the diameter is less than or equal to 4 inches, the radius must be less than or equal to 2 inches. The maximum volume occurs when the radius is 2 inches.

```

WINDOW
Xmin=-4.7
Xmax=4.7
Xscl=1
Ymin=-10
Ymax=80
Yscl=10
Xres=1
    
```



The maximum volume is approximately  $50.27 \text{ in}^3$ .



## Student Work Sample

The work on the next two pages was created by a group of students.

The solution illustrates many of the solution guide criteria. For example,

- Shows an understanding of the relationships among elements.

*The diagrams and explanations demonstrate the understanding of the relationships between the length and position of the straw and the radius length. These measurements are used to determine the height of the cylinder. The students' work shows that they know what measurements they must determine to find the volume of the cylinders.*

- Uses geometric and other mathematical principles to justify the reasoning used to analyze the problem.

*The students used the Pythagorean Theorem to compute the heights of the cylinder and the prism.*

- Communicates clear, detailed, and organized solution strategy.

*The students described what they were doing to solve the problem. The mathematical processes are detailed. They presented the equations in an organized and detailed manner. There is a clear step-by-step process showing the formulas they used. They wrote detailed statements to explain their work.*

## The Most Juice

1. First, we are given that we are looking for a juice container that holds the most juice. We know the container has to have a six-in. straw that can touch every point on the base with one inch remaining outside. Also, the hole that contains the straw must be 1 inch from the side of the container. Also, the base must be a square with dimensions of 4 x 4 or a circle that would be inscribed in that square. Finally, the containers could either be a rt. cylinder or a rectangular prism.

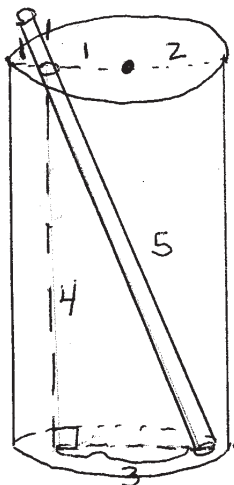
So we came up with these diagrams.

$$3^2 + x^2 = 5^2$$

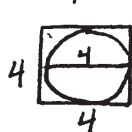
$$9 + x^2 = 25$$

$$x^2 = 16$$

$$x = 4$$



2. For a circle to be inscribed in a square its diameter must be the side of a square



the radius is also 2

$$3^2 + 3^2 = x^2$$

$$x = \sqrt{18}$$

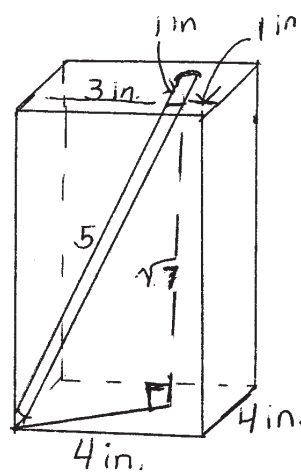
We use the Pythagorean Theorem to find the height.

$$\sqrt{18}^2 + x^2 = 5^2$$

$$18 + x^2 = 25$$

$$x^2 = 7$$

$$x = \sqrt{7}$$





3. Now, since the straw has to have at least 1 inch of it above the surface we know that the farthest reach the straw has in the cylinder must be 5 inches. Therefore we get a triangle with 3, 4, 5 in which four is the height.

3. cont... Now, since the straw has to have at least 1 inch of it above the surface we know that the farthest reach the straw has in the cylinder must be 5 inches. Therefore we get a right triangle with a height of  $\sqrt{7}$

4. Then we find the volume of each container.

Volume of cylinder

$$\begin{aligned} V &= bh \\ V &= \pi r^2 h \\ V &= \pi 2^2 (4) \\ V &= 16\pi \\ V &= 50.2 \text{ in}^3 \end{aligned}$$

Volume of Rectangular Prism

$$\begin{aligned} V &= bh \\ V &= 4^2(\sqrt{7}) \\ V &= 16(\sqrt{7}) \\ V &= 42.3 \text{ in}^3 \end{aligned}$$

5. In conclusion, we have concluded that the rt. cylinder would hold the most juice for the given specifications because it has more volume than the rectangular prism.

