Chapter 5:
Square Root Functions

I Can See Forever

Because you really enjoyed the balloon ride you took two years ago at the county fair, you have decided to take another one this year. This time, you take your binoculars with you to look around while you are airborne. As the balloon rises you use a device called a GPS (Global Positioning System) to determine your height (*h*) in meters and the distance (*d*) in kilometers to the farthest object that you can see. The table below compares these two measurements.

Height in meters (*h*)	15	30	45	60	75	90	105
Distance in kilometers (*d*)	13.833	19.562	23.959	27.665	30.931	33.883	36.598

1. Which is the independent variable? Explain your reason for making this decision. Make a scatterplot of the information in the table.

2. What parent function has the same approximate shape? Discuss the domain of the situation and how this will help you determine which function is most reasonable.

3. Determine a function that models this data.

4. If the balloon is 800 meters high, what is the farthest distance that you would be able to see?

5. A building is 16 kilometers away. To what height does the balloon have to rise for the building to be visible?

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus: (2A.9) Quadratic and square root functions. The student formulates equations and inequalities based on square root functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

(B) relate representations of square root functions, such as algebraic, tabular, graphical, and verbal descriptions.

(C) determine the reasonable domain and range values of square root functions, as well as interpret and determine the reasonableness of solutions to square root equations and inequalities.

(D) determine solutions of square root equations using graphs, tables, and algebraic methods.

(E) determine solutions of square root inequalities using graphs and tables.

(F) analyze situations modeled by square root functions, formulate equations or inequalities, select a method, and solve problems.

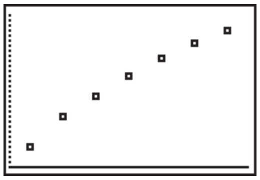
Scaffolding Questions:

- Do you think the balloon ride is in an area with lots of hills and mountains? Why?
- Why can't you "see forever?"
- Does there appear to be a strong pattern to the points in the plot?
- Looking at the numbers in the table, does the relationship appear to be linear? How would you check without looking at the graph?
- Sketch a parabola through the points to illustrate why, over a reasonably large domain, it would not be a good model.
- Models are built around assumptions. What assumptions are you making in this problem?
- When considering how high you have to be to see a building 800 meters away, does the height of the building matter?
- How large would the domain be if you were in a rocket? Would the range continue to follow this model?
- At what point do you think this relationship will no longer hold?
- Is this problem actually about inequalities?

Sample Solutions:

1. We would first have to be at a certain height and then find the distance to an object. The height is the independent variable since the distance you see is dependent on how high you are.

   The calculator graph is shown below:



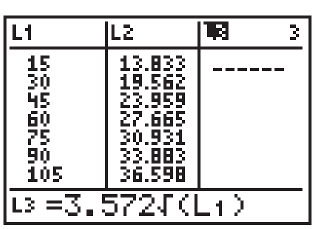
2. It might appear that a linear, quadratic, logarithmic, or square root function could be used as a model. The model is not linear because there does not appear to be a constant rate of change. The shape of the data does not match a quadratic function because the data appears to be concave down and not concave up. A logarithmic function would not be a good model because it would have negative values between zero and one and the graph begins to increase slowly. Thus, this function is not a good model. A square root function seems most reasonable because the shape of the data matches the shape of the graph of a square root function.

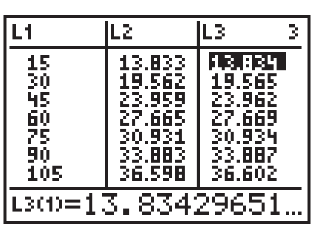
3. The general form of the square root function is $y = a\sqrt{x-h}+k$. The scatterplot indicates that there may not be horizontal or vertical shift, so an attempt to find an equation of the form $y = a\sqrt{x}$ is made. Since the dependent variable is distance, *d*, and the independent variable is height, *h*, the equation will be of the form $d = a\sqrt{h}$ The point (15, 13.8333) is used to determine a possible value for *a*.

$$d = a\sqrt{h}$$
$$13.833 = a\sqrt{15}$$
$$a = \frac{13.833}{\sqrt{15}}$$
$$a \approx 3.572$$

Checking another value for *h*, 75, gives $3.572\sqrt{75} = 30.934$, which is reasonably close to the value in the table, 30.931.

Using the list feature of the calculator to check shows that the model appears reasonably correct.





Additional Algebra II TEKS:
(2A.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

The student is expected to:

(A) identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations.

(B) collect and organize data, make and interpret scatterplots, fit the graph of a function to the data, interpret the results, and proceed to model, predict, and make decisions and critical judgments.

(2A.4) Algebra and geometry. The student connects algebraic and geometric representations of functions.

The student is expected to:

(A) identify and sketch graphs of parent functions, including linear ($f(x) = x$), quadratic ($f(x) = x^2$), exponential ($f(x) = a^x$), and logarithmic ($f(x) = \log_a x$) functions, absolute value of x ($f(x) = |x|$), square root

Notes

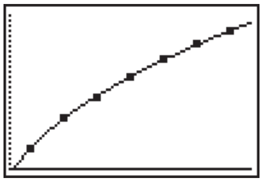
of *x* ($f(x)=\sqrt{x}$), and the reciprocal of *x* ($f(x) = 1/x$).

(2A.9) Quadratic and square root functions. The student formulates equations and inequalities based on square root functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

(E) determine solutions of square root inequalities using graphs and tables.

Checking by looking at a graph of $d = 3.572\sqrt{h}$ with the scatter plot shows a reasonable fit.



4. $d = 3.572\sqrt{800}$
   $d = 101.031$ km

5. $16 \le 3.572\sqrt{h}$
   $h \ge 20.064$ m

   You would have to be at least 20.064 m in the air.

Extension Questions:

- Does the domain and range that is appropriate for a rocket launch have the same values as the domain and range of the function you used to model the balloon ride?

  No. With a rocket your height could get exceptionally high but your view would be of half of the earth's surface, and going higher would not allow you to see any more of the earth's surface.

- Express your model as a function of distance not height.

$$d = 3.572\sqrt{h}$$
$$d^2 = 3.572^2 h$$
$$h = \frac{d^2}{3.572^2}$$

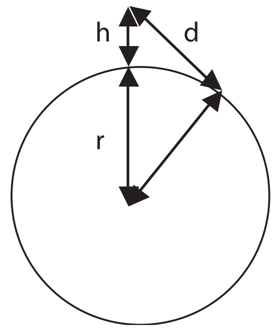
- Using the diagram, develop a model for this event in terms of *h* = height in meters, *d* = distance you can see in kilometers, and *r* = radius of the earth in kilometers. Let *r* = 6,378 km.

$$r^2 + d^2 = \left(\frac{h}{1000} + r\right)^2$$

$$r^2 + d^2 = \frac{h^2}{1{,}000{,}000} + \frac{hr}{500} + r^2$$

$$d^2 = \frac{h^2}{1{,}000{,}000} + \frac{hr}{500}$$

$$d = \sqrt{\frac{h^2}{1{,}000{,}000} + \frac{hr}{500}}$$



$$d = \sqrt{\frac{h^2}{1000000} + 12.756h}$$

- Use the model that was just developed to fill in the following table. How do these values compare with those originally given?

Distance (km)							
Height (m)	15	30	45	60	75	90	105

Distance (km)	13.833	19.562	23.959	27.665	30.931	33.883	36.598
Distance from original table	13.833	19.562	23.959	27.665	30.931	33.883	36.598
Height (m)	15	30	45	60	75	90	105

They are the same.

- Algebraically compare the model you developed in the original solution to the one you just developed. Where

Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

does the difference in estimation using the new one come from?

The original model $d = 3.572\sqrt{h} \Leftrightarrow d = \sqrt{12.759h}$ *should give estimates a little larger because the amount being added by the* $\frac{h^2}{1000000}$ *term in the new equation is negligible, and the 12.756h is less than 12.759h.*

I can see forever

1. x = height
y = distance
independent variable – height
the distance you can see depends on how high up you are.

2. square root function
The function has a starting point that goes in one direction to (literally) ∞, but for this equation, there has to be a stopping point because you cannot see forever

3. $d = 3.57\sqrt{h}$

$y = a\sqrt{k}$

$\frac{13.833}{\sqrt{15}} = \frac{a\sqrt{15}}{\sqrt{15}}$

$3.57 = a$

(3) To find the equation you substitute an x and y value from the chart into the equation $y = a\sqrt{k}$.

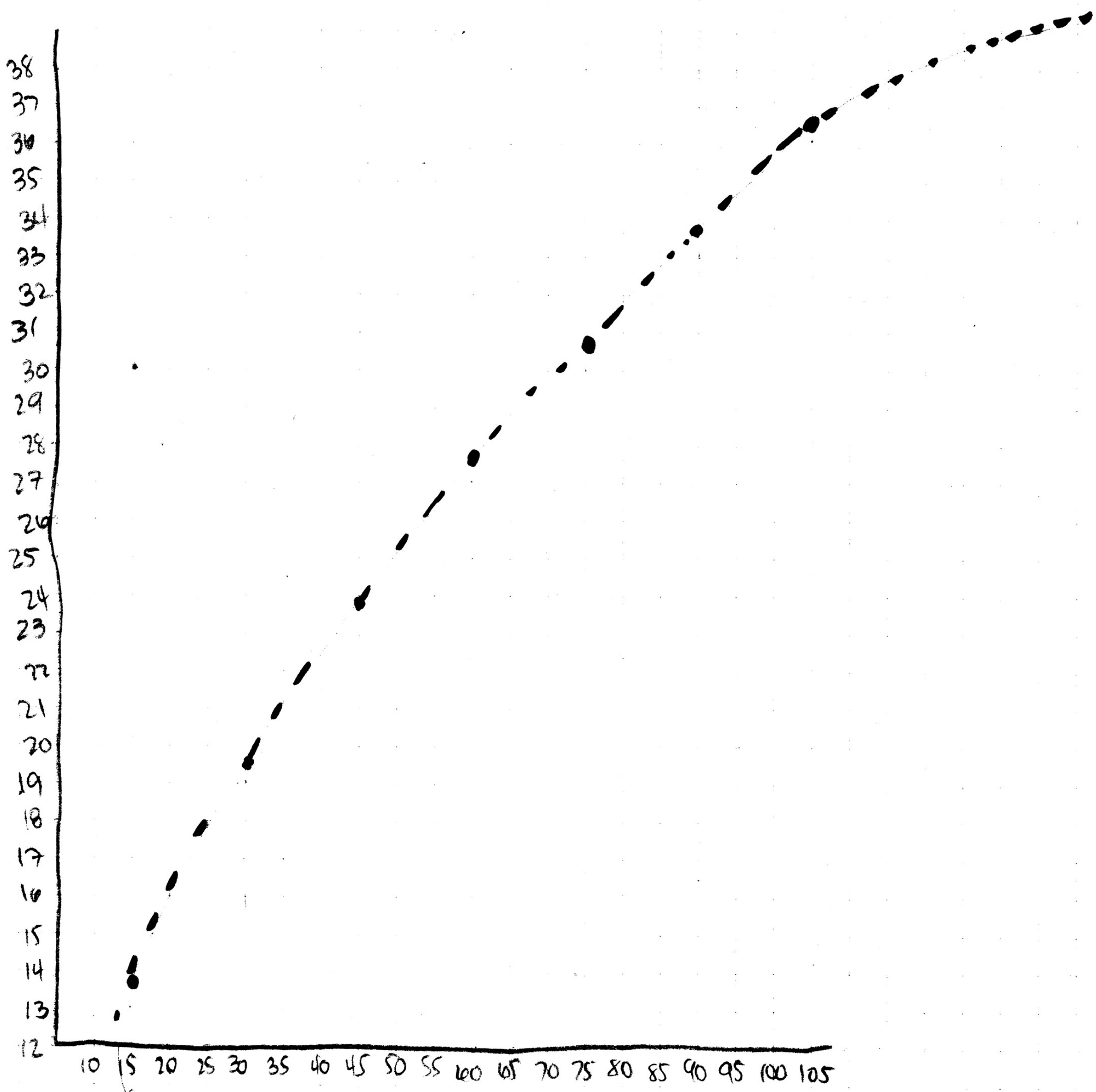
4. $d = 3.57\sqrt{800}$
$d = (3.57)(28.28)$
$d = 100.96$ km

(4) The question is asking about the distance (y value) so you put in the height (x) that they give you and you solve for y value.

5. $\frac{16}{3.57} = \frac{3.57\sqrt{h}}{3.57}$
$4.48^2 = \sqrt{h}^2$
$20.07\text{m} = h$

(5) The question is asking about height, and gives you distance, so you substitute in distance and you solve for height

6. $d \leq 3.57\sqrt{h}$
yes, an inequality can work because if you can only see 20 ft away, then you can also see things in front of 20 ft. For example, something at 10 ft or 5 ft.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I Was Going How Fast?

Accident investigators use the relationship $s = \sqrt{21d}$ to determine the approximate speed of a car, *s* mph, from a skid mark of length *d* feet, that it leaves during an emergency stop. This formula assumes a dry road surface and average tire wear.

1. A police officer investigating an accident finds a skid mark 115 feet long. Approximately how fast was the car going when the driver applied the brakes?

2. If a car is traveling at 60 mph and the driver applies the brakes in an emergency situation, how much distance does your model say is required for the car to come to a complete stop?

3. What is a realistic domain and range for this situation?

4. Does doubling the length of the skid double the speed the driver was going? Justify your response using tables, symbols, and graphs.

Notes

Materials:

Graphing calculator

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Additional Algebra II TEKS:

(2A.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

Scaffolding Questions:

- Would the relationship between a skid mark and the speed the car was going when it began to stop be a functional relationship?
- What factors influence realistic domain and range values for the situation?
- What is involved in stopping a car? Is it more than what a skid mark shows?
- Can a car stop without leaving a skid mark?
- Besides speed, what else would contribute to the length of a skid mark?
- Did the function that was given relating skid mark to speed take any other conditions into account?
- What might you change about the equation to take some of these other elements into account?

Sample Solutions:

1. The given value of 115 feet is the length of a skid mark, *d*. Substitute for *d* in the formula.

$$s = \sqrt{21d}$$
$$s = \sqrt{21 * 115}$$
$$s \approx 49 \text{ mph}$$

2. The given value of 60 mph is the approximate speed of the car, *s*. Substitute for *s* in the formula and solve for *d*.

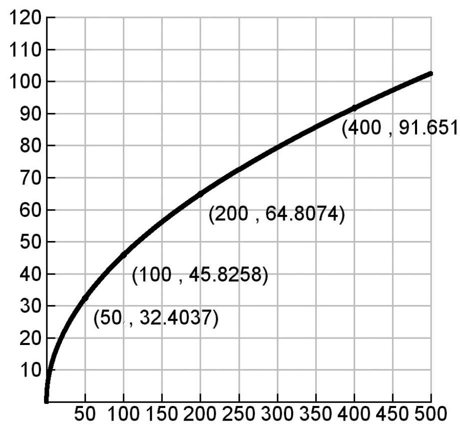
$$s = \sqrt{21d}$$
$$60 = \sqrt{21d}$$
$$3600 = 21d$$
$$d \approx 171 \text{ feet}$$

3. The function has a domain and a range of all positive real numbers. The situation dictates that there is some realistic maximum length and speed.

4. Doubling the length of the skid mark does not double the speed. The table below shows that when the skid value is doubled the speed value is not doubled. For example, 50 = 2(25) but 32.4 is not twice 22.9.

skid	25 ft	50 ft	100 ft	200 ft	400 ft
speed	22.9 mph	32.4 mph	45.8 mph	64.8 mph	91.7 mph

The graph also shows the function is not linear. Notice the labeled points.



To show this symbolically let *s'* represent the speed when the skid mark length is doubled.

$$s' = \sqrt{21*(2d)}$$
$$s' = \sqrt{2}\sqrt{21d}$$
$$s' = 1.414\sqrt{21d}$$

The speed is changed by a factor of approximately 1.414. It is not doubled.

Extension Questions:

- For what type of function would doubling the length of the *x*-value double the *y*-value?

  Linear functions of the form y = mx would be such that if x is doubled then y will be doubled.
  2y = m(2x)

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Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

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Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

- Express the length of a skid mark as a function of speed.

$$s = \sqrt{21d}$$
$$s^2 = 21d$$
$$d = \frac{s^2}{21}$$

Note that the square root function and this quadratic function are inverses of each other for positive values of s.

- Investigators find that a car that caused an accident left a skid mark 143 feet long. Damage to the car reveals that it was moving at a rate of 30 mph when it hit the other car. How fast was the car going when it started to skid?

If the skidding car had not been stopped by hitting another car, it would have needed another $\frac{30^2}{21} \approx 42.8$ *feet to skid before stopping. The total skid would have been (143 + 43) feet. A skid mark this long would imply the car was going* $\sqrt{21 * 186} \approx 62.5$ *mph when it started to skid.*

- What, besides the actual braking distance, do you think affects the total distance that it takes to stop a car in an emergency?

The distance that the car travels while the driver is reacting to the emergency situation needs to be added to the braking distance to determine the total stopping distance.

- There is a building on a corner of the highway that blocks a driver's view for 150 feet. If the speed limit on this stretch of highway is 55 mph, does a driver have enough time to stop if there is a car broken down in the highway 150 feet around the corner?

Determine the distance to stop at 55 mph hour using the rule.

$$s = \sqrt{21d}$$
$$55 = \sqrt{21d}$$
$$55^2 = 21d$$
$$d = \frac{3025}{21} \approx 144 \text{ feet}$$

Assuming the car could begin to stop the instant the driver saw the broken-down car, it would take 144 feet to stop from a speed of 55 mph. However, other factors that are not taken into account here, such as a driver's reaction time, would increase this stopping distance. Therefore, there is probably not enough room.

Tic Toc

There is a type of wall clock that keeps time by using weights, gears, and a pendulum. The pendulum swings back and forth to turn a series of wheels. As the wheels turn, the hands advance. The length of the pendulum determines how fast it swings. The faster the pendulum swings, the faster the clock goes.

Suppose your clock is running too slowly. As you attempt to fix your clock, you try different length pendulums. You create the following table by recording what you observe. The *period* of a pendulum is the length of time during which it swings from one side to the other and back again to the starting position.

Length of pendulum	10 cm	20 cm	30 cm	40 cm	50 cm	60 cm
Time of one complete swing	0.6 sec	0.9 sec	1.1 sec	1.3 sec	1.4 sec	1.6 sec

1. Create a scatterplot of this data with the length of the pendulum on the *x*-axis and the period in seconds on the *y*-axis.
2. Describe verbally the functional relationship between the length of a pendulum and its period.
3. Experiment with fitting various symbolic function rules to the scatterplot.
4. What would be a realistic domain for this situation?
5. In physics courses the following formula is derived that gives the period of the pendulum in seconds, *y*, in terms of the length in meters, *x,*

   $$y = 2\pi\sqrt{\frac{x}{(9.8)}}$$

   Graph this function with the data and determine if it is a reasonable model for this data.

6. Use your model to determine the length of a pendulum if the time to complete one cycle is 0.8 seconds.

7. From your observations and the manual that came with your clock, you realize that the period of the pendulum needs to be exactly 1 second. How long should the pendulum be for the clock to keep accurate time?

Notes

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Graphing calculator

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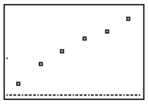
Additional Algebra II TEKS:
(2A.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

Scaffolding Questions:

- Where would data collection and analysis fit in this problem?
- Would you expect the period to be a function of the pendulum length?
- Suppose the relationship were not functional. What would that mean about the clock keeping the correct time?
- The function that models this situation has an infinite domain. Why is there a physical limit on the domain in this case?
- How do you change cm to m?
- The equation does not fit the data perfectly. What are some reasons why this would happen if it is the correct model?
- How do you know a quadratic function is not a reasonable model?
- Is the answer you found exact?
- If you do not have an exact answer, will the clock keep the exact time?

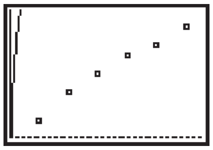
Sample Solutions:

1. The scatterplot is created with the horizontal axis representing length in centimeters and the vertical axis representing time in seconds.



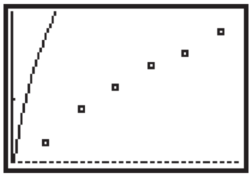
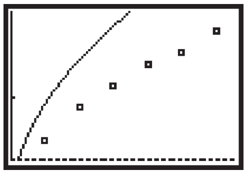
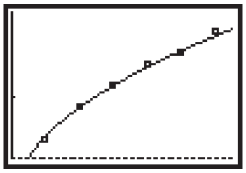
2. The time it takes to complete one swing of the pendulum is dependent on its length. As the length increases, the time increases.

3. Because the data plot is an increasing graph that is curved down, the parent function $y = \sqrt{x}$ was tried.



It did not fit the function, but various multiples were tried.

$y = 0.5\sqrt{x}$ $y = 0.3\sqrt{x}$ $y = 0.2\sqrt{x}$

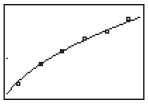
$y = 0.2\sqrt{x}$ closely models the data.

4. From 0 to the height of the clock from the floor.

5. To use this model with the data plotted in centimeters, the *x*-value must be converted to meters.

$$x \text{ cm} \bullet \frac{1 \text{ meter}}{100 \text{ cm}}$$

$$y = 2\pi\sqrt{\frac{x}{100(9.8)}}$$



The student is expected to:

(B) collect and organize data, make and interpret scatterplots, fit the graph of a function to the data, interpret the results, and proceed to model, predict, and make decisions and critical judgments.

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The student is expected to:

(E) determine solutions of square root inequalities using graphs and tables

Connection to TAKS:

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6. From the table for the model function, $y = 2\pi\sqrt{\dfrac{x}{100(9.8)}}$, the value at 0.8 seconds is 15.9 cm.

X	Y1	
15.8	.7978	
15.9	.80032	
16	.80284	
16.1	.80534	
16.2	.80784	
16.3	.81033	
16.4	.81281	

X=15.9

One can also solve using the formula

$$0.8 = 2\pi\sqrt{\frac{x}{100(9.8)}}$$

$$\frac{0.8}{2\pi} = \sqrt{\frac{x}{100(9.8)}}$$

$$\left(\frac{0.8}{2\pi}\right)^2 = \frac{x}{100(9.8)}$$

$$x = 100(9.8)\left(\frac{0.8}{2\pi}\right)^2 \approx 15.9 \text{ cm.}$$

7. Consider the original function given with *x* in meters.

$$y = 2\pi\sqrt{\frac{x}{9.8}}$$

$$1 = 2\pi\sqrt{\frac{x}{9.8}}$$

$$1 = 2.007\sqrt{x}$$

$$\frac{1}{2.007} = \sqrt{x}$$

$$x = 0.248 \text{ m}$$

$$x = \frac{0.248 \text{ m}}{1} \bullet \frac{100 \text{ cm}}{1 \text{ m}} = 24.8 \text{ cm}$$

Extension Questions:

- If each complete swing of the pendulum moves the second hand one unit, and the pendulum is set to a length of 24.8 cm, will the clock run fast? If so, how fast by the end of one day?

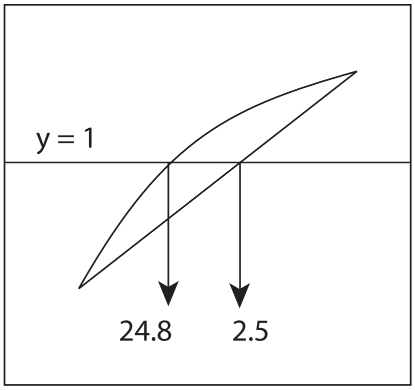
$$2\pi\sqrt{\frac{0.248}{9.8}} = 0.9995227211 \text{ sec}$$

*Thus, each swing will be (1-0.9995227211) seconds = 0.0004772789037 seconds too fast. In a minute, 60 swings, it will be 0.0286367342 seconds too fast. It would be 0.0286367342*60*24 = 41.23689728 seconds too fast a day. It would be over 4 hours off a year.*

- Explain why a quadratic function would not appear to model the situation.

In the quadratic function, as x increases y also increases until x reaches the vertex of the function. After that value, as x increases the value of y decreases. As long as the pendulum does not hit the floor, as the pendulum length increases the period always increases. Thus, although the points that were plotted might appear quadratic for the domain of the situation, it would not be a reasonable model.

- Knowing that the pendulum needed a period of 1 second, you could place the proper length between 20 cm and 30 cm. If you assumed a linear relationship between those two points, would you have made your pendulum too long or too short? Illustrate your answer with a graph and then algebraically.



The pendulum will be too long.

y = 2x + 0.5 is the equation of the line through the points (0.2, 0.9) and (0.3, 1.1). Solving this when y = 1 gives x = 25 cm. Using the curve we previously found x = 24.8 cm. The pendulum would be too long.

- Each time the pendulum makes a complete swing it advances a gear one tooth. The gear has 47 teeth. When the gear makes one complete revolution, the minute hand advances 1 minute. What is the length of the pendulum necessary for the clock to keep accurate time?

  The pendulum needs to make 47 complete trips a minute. The period must be $\frac{60}{47}$ *seconds.*

$$\frac{60}{47} = 2\pi = \left(\frac{x}{9.8}\right)$$

$$\frac{60}{47} \bullet \frac{1}{2\pi} = \sqrt{\frac{x}{9.8}}$$

$$\frac{x}{9.8} = \left(\frac{60}{47} \bullet \frac{1}{2\pi}\right)^2$$

$$x = 9.8\left(\frac{60}{47} \bullet \frac{1}{2\pi}\right)^2$$

$$x \approx 0.4046 \text{ m or } 40.46 \text{ cm}$$

- Express the function in the form $y = a\sqrt{x}$.

$$2\pi\frac{\sqrt{x}}{\sqrt{9.8}} = \frac{2\pi}{\sqrt{9.8}}\sqrt{x} \approx 2.007\sqrt{x}$$

Chapter 6:
Exponential and Logarithmic Functions

Desert Bighorn Sheep

Among the many species that have been endangered at one time or another is the desert bighorn sheep. The desert bighorn sheep is important to preserve because it is sensitive to human-induced problems in the environment and is a good indicator of land health.

Desert Bighorn Ram
Courtesy of Jeff Heinatz, Photographer. www.llnet.net/~heinatz

It is estimated that in the 1600s, there were about 1.75 million bighorn sheep in North America. By 1960, the bighorn sheep population had dropped to about 17,000.

Wildlife biologists have data showing that in 1880, there were around 1,500 bighorn sheep in west Texas. By 1955, the population had dwindled to 25.

Efforts to reintroduce desert bighorn sheep in Texas began around 1957. By 1993, there were about 400 desert bighorn sheep roaming free or in captivity.

1. Assume that, from 1880 to 1955, the annual percentage decrease in the bighorn sheep population was fairly constant. Model the population with an exponential function, $P = ab^t$, where t is the number of years since 1880, a is the population of the bighorn sheep in west Texas in 1880, and P is the annual population in west Texas.

2. Assume that, starting in 1957 when reintroduction began, the annual percentage increase in the bighorn sheep population was fairly constant.

Model the population with an exponential function, $P = ab^t$, where t is the number of years since 1957, P is the annual population in west Texas, and a is the population in 1957.

3. Describe mathematical domain and range values for these two functions. Describe reasonable domain and range values for the situation.

4. From 1880 to 1955, by what rate was the population decreasing? From 1957 to 1993, by what rate was the population increasing?

5. By what year had the sheep population dropped to 750 or less? Use technology (tables and/or graphs) and algebraic methods to determine this.

6. If the reintroduction program continues, in what year will the bighorn sheep population again be at least 750? Use technology (tables and/or graphs) and algebraic methods to determine this.

7. In 2001, it was reported that there were then 500 bighorn sheep in Texas. Given the reintroduction model, is this reasonable? Why or why not?

Notes

References used in this section:

Henry E. McCutchen, Desert Bighorn Sheep, http://biology.usgs.gov/s+t/noframe/r039.htm

Texas Parks and Wildlife, Desert Bighorns on the Rise, www.biggamehunt.net/sections/Texas/Desert_Bighorns_on_the_Rise_12170112.html

Materials:

Graphing calculator

Algebra II TEKS Focus: (2A.11) Exponential and logarithmic functions. The student formulates equations and inequalities based on exponential and logarithmic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

(C) determine the reasonable domain and range values of exponential and logarithmic functions, as well as interpret and determine the reasonableness of solutions to exponential and logarithmic equations and inequalities.

Scaffolding Questions:

- What are the independent and dependent variables in this situation?
- How can you count the years to make the number computation easier?
- What is the initial condition in each model?
- What type of equation must you solve to find the bases?
- How can you use logarithms to help you find an unknown exponent in an exponential equation?
- What determines the reasonable domain and range for each situation?

Sample Solutions:

1. The independent variable for the situation is time, *t*, in years. Let 1880 correspond to time *0*. Then 1955 will correspond to time *75*. The data points are (0, 1,500) and (75, 25).

   The general model is $P = ab^t$. Substituting 0 for *t* and 1,500 for *P* gives *a* = 1,500.

   Use the point (75, 25). Substitute 75 for *t* and 25 for *P*.

$$1{,}500b^{75} = 25$$
$$b^{75} = 0.0167$$
$$b = (0.0167)^{1/75}$$
$$b = 0.947$$

   The model for the decreasing population is $P = 1{,}500(0.947)^t$.

2. For the reintroduction model, let 1957 correspond to time *0*. Then 1993 will correspond to time 36.

   We need to apply the model from problem 1 to determine the population in 1957.

$$\begin{aligned} P &= 1{,}500(0.947)^t \\ &= 1{,}500(0.947)^{77} \\ &= 22.649 \\ &\approx 23 \end{aligned}$$

The data points are now (0, 23) and (36, 400). Substituting (0, 23) in the general model gives $a = 23$.

Now substitute (36, 400) in $P = 23b^t$ and interchange sides to get

$$\begin{aligned} 23b^{36} &= 400 \\ b^{36} &= 17.39 \\ b &= 17.39^{1/36} \\ b &\approx 1.083 \end{aligned}$$

The model for the growing population is $P = 23(1.083)^t$.

3. Since these are exponential models, the mathematical domain for both models is the set of all real numbers, and the range is the set of all positive real numbers.

   The domain for the decreasing population model is the set of integers from 0 to 77 inclusive (corresponding to the years from 1880 to 1955). The range is the set {1,500, 1,420, 1,345,..., 25} where, in a table of values, we are rounding down to the nearest sheep in the annual count.

   The domain for the increasing population model is the non-negative integers with $t = 0$ corresponding to 1957. Since the reintroduction project continues, there appears to be no upper bound on the domain. The range is the set {23, 24, 26, 29,...}.

   The domain, and, therefore, the range will be restricted by practical concerns such as space and available food and water for the sheep.

4. In the decreasing population model, $b = 0.947$, the

(D) determine solutions of exponential and logarithmic equations using graphs, tables, and algebraic methods.

(F) analyze a situation modeled by an exponential function, formulate an equation or inequality, and solve the problem.

Notes

Additional Algebra II TEKS: (2A.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

The student is expected to:

(A) identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations.

(2A.9) Quadratic and square root functions. The student formulates equations and inequalities based on square root functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

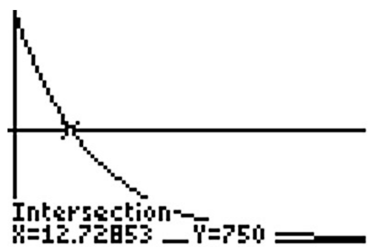
The student is expected to:

(E) determine solutions of square root inequalities using graphs and tables.

annual percentage decrease is about 5.3% because 1 – 0.947 = 0.053.

In the growing population model, $b = 1.0825$, the annual percentage increase is about 8.3% because 1.083 = 1 + 0.083.

5. To determine how many years it takes the decreasing population to drop to at most 750 sheep, we use the calculator's graph or table functions. Let $Y_1 = 1500(0.947)^x$ and $Y_2 = 750$. Graph the functions and find the point of intersection.



X	Y1	
11	824.03	
12	780.35	
13	738.99	
14	699.83	
15	662.74	
16	627.61	
17	594.35	

Y1=1500*.947^X

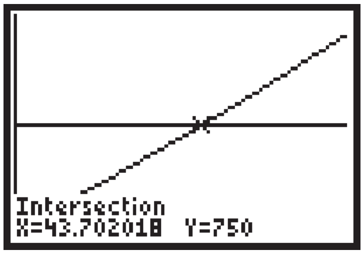
The graph shows that it takes nearly 13 years for the population to decrease to 750 sheep. The table shows that in 12 years, the population has dropped to 780 sheep, and in 13 years, it has dropped to 738.

To see this algebraically, we must solve the equation

$$
\begin{aligned}
1500(0.947)^t &= 750 \\
(0.947)^t &= 0.5 \\
t &= \frac{\ln(0.5)}{\ln(0.947)} \\
t &\approx 12.729
\end{aligned}
$$

Towards the end of 1893 (since 1880 + 13 = 1893), the sheep population will be about 750. From that point on, it will be less since the population is decreasing.

6. We use the same procedure to determine when the increasing population will again reach 750 sheep. Let $Y_2 = 23(1.083)^x$ and $Y_3 = 750$.



X	Y2	Y3
42	654.82	750
43	709.17	750
44	768.03	750
45	831.78	750
46	900.82	750
47	975.59	750
48	1056.6	750

X=44

It will take 44 years after 1957 for the sheep population to reach 750.

To show this algebraically, we solve the equation

$$23(1.083)^t = 750$$
$$1.083^t = 32.609$$
$$t = \frac{\ln(32.609)}{\ln(1.083)}$$
$$t \approx 43.702$$

During 2001 (since 1957 + 44 = 2001), the population should become 750 again and then continue to increase.

7. The model for the growing population shows that in 2001, there should be about 768 bighorn sheep. The data said there were only 500. The model assumes that annual growth occurs at a constant rate. This assumption is probably not realistic for this situation because a number of factors can affect the size of the sheep population—weather, disease, predators, food and water supply, etc.

Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

Extension Questions:

- Why are exponential functions used to model the decline and the growth in desert bighorn sheep populations?

  Experience shows that these models fit data well. Annual data describes the population as a percentage of the previous year's population. This is similar to interest earned/interest paid, which is modeled by

  $$A = P(1 \pm r)^t$$

  where r is the percent interest and t is the number of years.

  In our functions, $b = 1 \pm r$.

- In problem 5, you found that the 1880 population of 1,500 bighorn sheep declined to one-half its size in 12.7 years. In other words, its half-life was 12.7 years. Is the half-life of a declining population dependent on its original size?

  To investigate this we could use an arbitrary initial population size. Let C = the original population size.

  $$\frac{C}{2} = C(0.947^t)$$
  $$0.947^t = \frac{1}{2}$$

  The value of t does not depend upon the choice of C.

  This shows the half-life, 12.7 years, is independent of the original population size.

- Suppose that with the reintroduction program, wildlife experts predict that the bighorn sheep population will increase by 5% to 8% per year. For this range of percentage increase, how many years will it take the population to double?

  The population at the time of reintroduction was 23 sheep, so we want to know in how many years will there be 56 sheep. We can solve the following equation for t in terms of r. Then we can let r range from 5% to 8% and determine t.

$$
\begin{aligned}
23(1+r)^t &= 46 \\
(1+r)^t &= 2 \\
t\ln(1+r) &= \ln(2) \\
t &= \frac{\ln(2)}{\ln(1+r)}
\end{aligned}
$$

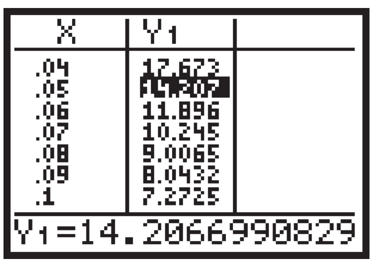
$$r = 5\% = 0.05 \Rightarrow t = \frac{\ln(2)}{\ln(1.05)} = 14.207$$

$$r = 8\% = 0.08 \Rightarrow t = \frac{\ln(2)}{\ln(1.08)} = 9.006$$

As r increases from 5% to 8%, the number of years it takes the sheep population to double decreases from 14 years to 9 years.

We can also use the calculator's table function to determine doubling times.

Let $Y_1 = \frac{\ln(2)}{\ln(1+X)}$ *where X = percentage increase in population, starting with X = 0.05. The results are shown below.*



It will take between 9 and 14 years for the sheep population to double with population increase rates of 5% to 8%.

Comparing an Exponential Function and Its Inverse

Two friends, Emily and Lorraine, were working on an algebra task assigned to their team. They were arguing about the meaning of the inverse of a function. They consulted a dictionary and found the following definitions for *inverse*:

1. reversed in position, direction, or tendency,
2. opposite to in nature or effect,
3. inverted or turned upside down.

Here is their assignment:

Analyze and compare the functions below.

$$Y_1 = a^x \qquad Y_3 = \log_a(x) = \frac{\ln(x)}{\ln(a)}$$

$$Y_2 = \left(\frac{1}{a}\right)^x \qquad Y_4 = \log_{\frac{1}{a}}(x) = \frac{\ln(x)}{\ln\left(\frac{1}{a}\right)}$$

To help in the comparisons, they have been asked to complete three sets of activity sheets in which a is given three different values, 2, 10, and e.

After they completed the activity sheets, Emily claimed that the pairs of functions that are inverses of each other are

Y_1 and Y_2

Y_3 and Y_4

Lorraine claimed that the pairs of functions that are inverses of each other are

Y_1 and Y_3

Y_2 and Y_4

Complete their assignment to determine which student is correct.

Comparing an Exponential Function and Its Inverse: Set A

Function	Graph or table	Domain	Range	Intercepts
$Y_1 = 2^x$				
$Y_2 = \left(\frac{1}{2}\right)^x$				
$Y_3 = \log_2(x)$				
$Y_4 = \log_{\frac{1}{2}}(x)$				

Comparing an Exponential Function and Its Inverse: Set A

Function	Increases or decreases?	Concave up or down?	Transformation on Y_1
$Y_1 = 2^x$			parent function
$Y_2 = \left(\frac{1}{2}\right)^x$			
$Y_3 = \log_2(x)$			
$Y_4 = \log_{\frac{1}{2}}(x)$			

Which pairs of functions are inverses of each other?

Comparing an Exponential Function and Its Inverse: Set B

Function	Graph or table	Domain	Range	Intercepts
$Y_1 = 10^x$				
$Y_2 = \left(\frac{1}{10}\right)^x$				
$Y_3 = \text{Log}(x)$				
$Y_4 = \log_{\frac{1}{10}}(x)$				

Comparing an Exponential Function and Its Inverse: Set B

Function	Increases or decreases?	Concave up or down?	Transformation on Y_1
$Y_1 = 10^x$			parent function
$Y_2 = \left(\frac{1}{10}\right)^x$			
$Y_3 = \text{Log}(x)$			
$Y_4 = \log_{\frac{1}{10}}(x)$			

Which pairs of functions are inverses of each other?

Comparing an Exponential Function and Its Inverse: Set C

Function	Graph or table	Domain	Range	Intercepts
$Y_1 = e^x$				
$Y_2 = \left(\frac{1}{e}\right)^x$				
$Y_3 = \log_e(x)$				
$Y_4 = \log_{\frac{1}{e}}(x)$				

Comparing an Exponential Function and Its Inverse: Set C

Function	Increases or decreases?	Concave up or down?	Transformation on Y_1
$Y_1 = e^x$			parent function
$Y_2 = \left(\frac{1}{e}\right)^x$			
$Y_3 = \log_e(x)$			
$Y_4 = \log_{\frac{1}{e}}(x)$			

Which pairs of functions are inverses of each other?

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus:
(2A.4) Algebra and geometry. The student connects algebraic and geometric representations of functions.

The student is expected to:

(C) describe and analyze the relationship between a function and its inverse.

(2A.11) Exponential and logarithmic functions. The student formulates equations and inequalities based on exponential and logarithmic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

(A) develop the definition of logarithms by exploring and describing the relationship between exponential functions and their inverses.

Scaffolding Questions:

- Think about the earlier function families we have studied. Describe some inverse functions in these families. Try these:

  What is the inverse of $y = 2x + 3$?

  What is the inverse of $y = (x + 1)^2$ where $x \geq -1$?

  What is the inverse of $y = 2(x - 1)^3$?

- Does every function have an inverse that is also a function?
- What must be true about a function for it to have an inverse that is also a function?
- How can you use the graph of a function to find the graph of its inverse?
- What is true about the domain and range of the inverse of a function?
- If the point (1, 2) is on the graph of a function, what corresponding point is on the graph of its inverse?
- With an exponential function, we input an exponent and output a power. What is true about a logarithmic function?

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Additional Algebra II TEKS:
(2A.4) Algebra and geometry. The student connects algebraic and geometric representations of functions.

The student is expected to:

(A) identify and sketch graphs of parent functions, including linear ($f(x) = x$), quadratic ($f(x) = x^2$), exponential ($f(x) = a^x$), and logarithmic ($f(x) = \log_a x$) functions, absolute value of x ($f(x) = |x|$), square root of x ($f(x) = \sqrt{x}$), and reciprocal of x ($f(x) = 1/x$).

Connection to TAKS:

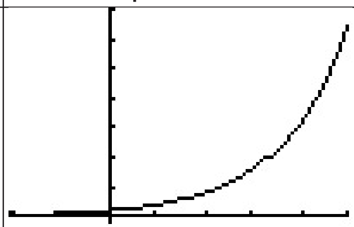
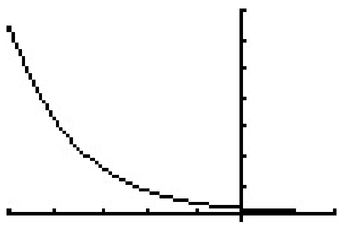
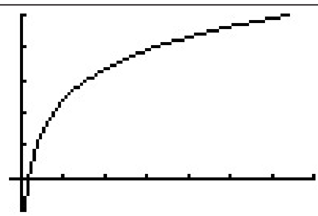
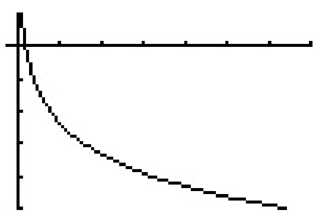
Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Sample Solutions:

Comparing an Exponential Function and Its Inverse: Set A

Function	Graph or table	Domain	Range	Intercepts
$Y_1 = 2^x$		x can be any real number.	y can be any positive real number.	x-intercept: none y-intercept: (0,1)
$Y_2 = \left(\frac{1}{2}\right)^x$		x can be any real number.	y can be any positive real number.	x-intercept: none y-intercept: (0,1)
$Y_3 = \log_2(x)$		x can be any positive real number.	y can be any real number.	x-intercept: (1,0) y-intercept: none
$Y_4 = \log_{\frac{1}{2}}(x)$		x can be any postive real number.	y can be any real number.	x-intercept: (1,0) y-intercept: none

Comparing an Exponential Function and Its Inverse: Set A

Function	Increases or decreases?	Concave up or down?	Transformation on Y_1
$Y_1 = 2^x$	Increases	Concave up	parent function
$Y_2 = \left(\frac{1}{2}\right)^x$	Decreases	Concave up	Reflect $Y_1 = 2^X$ over y-axis
$Y_3 = \log_2(x)$	Increases	Concave down	Reflect $Y_1 = 2^X$ over $y = x$.
$Y_4 = \log_{\frac{1}{2}}(x)$	Decreases	Concave up	Reflect $Y_1 = 2^X$ over y–axis. Then reflect over $y = x$.

The inverse function pairs are Y_1 and Y_3 and also Y_2 and Y_4. For each pair, the graph of one function is the reflection of the graph of the other over the line $y = x$.

Comparing an Exponential Function and Its Inverse: Set B

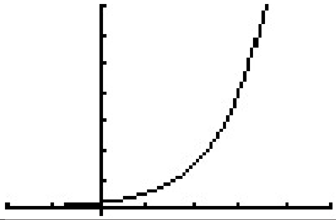
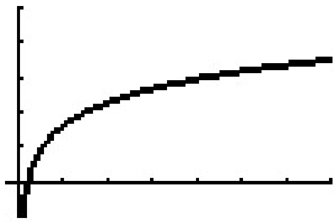
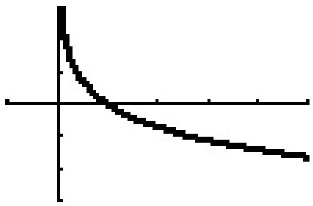
Function	Graph or table	Domain	Range	Intercepts
$Y_1 = 10^x$	X, Y1 -3, .001 -2, .01 -1, .1 0, 1 1, 10 2, 100 3, 1000 Y1=10^X	*x* can be any real number.	*y* can be any positive real number.	*x*-intercept: none *y*-intercept: (0,1)
$Y_2 = \left(\frac{1}{10}\right)^x$	X, Y2 -3, 1000 -2, 100 -1, 10 0, 1 1, .1 2, .01 3, .001 Y2=(1/10)^X	*x* can be any real number.	*y* can be any positive real number.	*x*-intercept: none *y*-intercept: (0,1)
$Y_3 = \text{Log}(x)$	X, Y3 0, ERROR 10, 1 100, 2 1000, 3 10000, 4 100000, 5 1E6, 6 Y3=log(X)	*x* can be any positive real number.	*y* can be any real number.	*x*-intercept: (1,0) *y*-intercept: none
$Y_4 = \log_{\frac{1}{10}}(x)$	X, Y4 0, ERROR 10, -1 100, -2 1000, -3 10000, -4 100000, -5 1E6, -6 Y4=(log(X))/log…	*x* can be any postive real number.	*y* can be any positive real number.	*x*-intercept: (1,0) *y*-intercept: none

Comparing an Exponential Function and Its Inverse: Set B

Function	Increases or decreases?	Concave up or down?	Transformation on Y_1
$Y_1 = 10^x$	Increases	Concave up. Range values increase faster and faster.	parent function
$Y_2 = \left(\frac{1}{10}\right)^x$	Decreases	Concave up. Range values increase faster and faster.	Reflect Y_1 over *y*–axis. $f(-x) = f(x)$
$Y_3 = \text{Log}(x)$	Increases	Concave down. Range values increase more and more slowly.	Reflect Y_1 over $y = x$. If (a,b) is on graph of Y_1, then (b,a) is on graph of Y_3.
$Y_4 = \log_{\frac{1}{10}}(x)$	Decreases	Concave up. Range values increase more and more slowly.	Reflect over *y*-axis. Then reflect over $y=x$. If (a,b) is on graph of Y_1, then $(b,-a)$ is on graph of Y_3.

The inverse function pairs are Y_1 and Y_3 and also Y_2 and Y_4. For each pair, the input (domain) values and output (range) values of one function become the output values and input values of the other, respectively.

Comparing an Exponential Function and Its Inverse: Set C

Function	Graph or table	Domain	Range	Intercepts
$Y_1 = e^x$		x is any real number.	y is any positive real number.	x-intercept: none y-intercept: (0,1)
$Y_2 = \left(\frac{1}{e}\right)^x$		x is any real number.	y is any positive real number.	x-intercept: none y-intercept: (0,1)
$Y_3 = \log_e(x)$		x is any positive real number.	y is any real number.	x-intercept: (1,0) y-intercept: none
$Y_4 = \log_{\frac{1}{e}}(x)$		x is any positive real number.	y is any real number.	x-intercept: (1,0) y-intercept: none

Comparing an Exponential Function and Its Inverse: Set C

Function	Increases or decreases?	Concave up or down?	Transformation on Y_1
$Y_1 = e^x$	Increases	Concave up	parent function
$Y_2 = \left(\frac{1}{e}\right)^x$	Decreases	Concave up	Reflect over Y_1 y-axis.
$Y_3 = \log_e(x)$	Increases	Concave down	Reflect Y_1 over $y = x$.
$Y_4 = \log_{\frac{1}{e}}(x)$	Decreases	Concave up	Reflect Y_1 over y-axis. Then reflect over $y = x$.

The inverse function pairs are Y_1 and Y_3 and also Y_2 and Y_4. For each pair, the graph of one function is the reflection of the graph of the other over the line $y = x$.

For each set the results are the same. The inverse function pairs are Y_1 and Y_3 and also Y_2 and Y_4. Lorraine appears to be correct.

Extension Questions:

- Why do you suppose someone might think that $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$ are inverse functions?

  *Lots of people think of 2 and $\frac{1}{2}$ as **inverses** because they are multiplicative inverses, or you invert 2 to get $\frac{1}{2}$.*

- What is a commonsense way of describing the inverse of a function?

  If a function describes a sequence of operations on an input value to get its output value, the inverse of the function undoes those operations in reverse order.

- Does every exponential function have an inverse that is a function? Why or why not?

  Yes. Any exponential function is a one-to-one function. Each output value comes from exactly one input value. If a function is one-to-one, it has an inverse that is a function.

- If $f(x) = a^x$, then $f^{-1}(x) = \log_a(x)$. What can you tell me about the domain and range of f^{-1}?

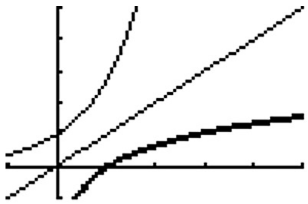
  The domain of f becomes the range of f^{-1}, and the range of f becomes the domain of f^{-1}.

- For example, what happens if (a, b) is on the graph of f? What happens to intercepts?

  If (a, b) is on the graph of f, (b, a) is on the graph of f^{-1}. Y-intercepts on the graph of f become x-intercepts on the graph of f^{-1}, and vice versa.

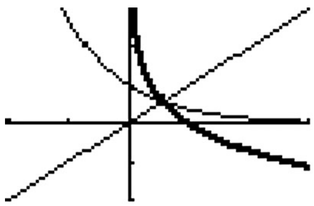
- What does all this mean geometrically? How can you use transformations to find the graph of the logarithmic function that is the inverse of an exponential function?

Geometrically, you are switching input and output values point by point. To do this with transformations, you reflect the graph of the exponential function over the line y = x to get the graph of its inverse, a logarithmic function.



$Y_1 = 2^x$

$Y_3 = \log_2(x)$



$Y_2 = \left(\frac{1}{2}\right)^x$

$Y_4 = \log_{\frac{1}{2}}(x)$

- The team that had the set involving exponential and logarithmic functions, base 10, used tables instead of graphs. Why would they do this?

Powers of 10 grow big quickly, making it harder to analyze the exponential function and logarithmic function graphically. For this set, you can see patterns more easily in tables.

- If we know how an exponential function, *f*, behaves, what other observations can we make about its logarithmic function, f^{-1}?

If an exponential function is a growth (increasing) function, so is its inverse function.

If an exponential function is a decay (decreasing) function, so is its inverse function.

Exponential functions are concave up. Logarithmic functions are concave down.

For example, in a growth situation with an exponential function, as input (an exponent) x increases, the output (power of x) values increase more rapidly. With a logarithmic function, as input (power of) x increases, the output (exponent) values increase but more slowly.

Saving Money, Making Money

Suppose you receive for graduation a gift of $1,200 from your favorite relative. You are required to invest at least $800 of the gift in a no-withdrawal savings program for at least two years. You have designed two plans to consider.

Plan A: *First Savings Bank (FSB)* pays 6% interest, compounded annually on savings accounts. *Employee's Credit Union (ECU)* has options that allow you to choose your interest rate and how often your interest is compounded.

1. Determine how much you would have at the end of 2 years if you decided to invest $1,000 at *FSB.*

2. One of the options at *ECU* pays 5% interest annually and compounds interest quarterly. How much would your initial deposit there need to be to have the same amount that you would have after investing $1,000 with *FSB* for 2 years?

3. Another option at *ECU* compounds interest monthly. If you invest $1,000 compounded monthly with ECU, what interest rate would they have to pay for you to have the same amount that you would have after investing $1,000 with *FSB* for 2 years?

4. If one plan at *ECU* pays 4.75% interest compounded bimonthly (every two months), and you invest $1,000 in that plan, how long would it take for you to have the same amount that you would have after investing $1,000 with *FSB* for 2 years?

Plan B: *ECU* also has some plans in which interest is compounded continuously. You are still comparing with an investment of $1,000 at 6% annual interest at *FSB*.

1. Suppose your savings will earn 5% interest compounded continuously at *ECU*. How much would your initial deposit there need to be to have the amount you could have with *FSB* in 2 years?

2. If your initial deposit at *ECU* is $1,000, what continuous compound

interest rate would *ECU* need to pay for you to have the amount you could have with *FSB* in 2 years?

3. You speculate about making no additional deposits and no withdrawals from the savings account for 10 years. What continuous compound interest rate would *ECU* have to pay so that your initial $1,000 doubles in 10 years? How would that compare with the amount in your account at *FSB* after 10 years?

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus: (2A.11) Exponential and logarithmic functions. The student formulates equations and inequalities based on exponential and logarithmic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

(D) determine solutions of exponential and logarithmic equations using graphs, tables, and algebraic methods.

(F) analyze a situation modeled by an exponential function, formulate an equation or inequality, and solve the problem.

Additional Algebra II TEKS: (2A.2) Foundations for functions. The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

The student is expected to:

(A) use tools including factoring and properties of exponents to simplify

Scaffolding Questions:

- What function rule expresses the amount of money, *A* dollars, in a savings account as a function of the number of years, *t*, money is in the account if interest is compounded *n* times per year?
- What function rule do you use if interest is compounded continuously?
- As you solve for different parameters in these functions what types of equations do you encounter?
- What methods do you have for solving exponential equations?
- What methods seem to work best in these situations?

Sample Solutions:

Plan A

1. For *FSB*, the function expressing the amount of money, *A* dollars, in an account in terms of years of deposit, *t* years, is

$$A = P(1 + r)^t$$

where *P* = the initial deposit (principal) in dollars, and *r* = the annual interest rate.

After 2 years, the amount in an account with $1,000 principal and 6% interest will be $1{,}000(1 + .06)^2 = 1{,}000(1.06)^2 = \$1{,}123.60$.

2. If *ECU* pays 5% interest annually and compounds quarterly, the function expressing *A* in terms of years of deposit, *t*, is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where *n* is the number of compounding periods per year.

In this problem, $r = 0.05$, $n = 4$, and $t = 2$. We need to determine P so that we break even with *FSB.* Therefore we need to solve

$$P\left(1+\frac{0.05}{4}\right)^{4*2} = 1{,}123.60$$
$$P(1.0125)^8 = 1{,}123.60$$
$$1.104486P = 1{,}123.60$$
$$P = \frac{1{,}123.60}{1.104486}$$
$$= 1{,}017.3057$$

Our initial deposit at *ECU* would need to be \$17.31 more than our deposit at *FSB* for us to have the same amount in 2 years.

3. This time we know that at *ECU*, $P = 1{,}000$, $n = 12$, and $t = 2$. We need to determine the interest rate, r, so that we have the same amount as we would have with *FSB*. Therefore, we need to solve

$$1{,}000\left(1+\frac{r}{12}\right)^{12*2} = 1{,}123.6$$
$$\left(1+\frac{r}{12}\right)^{24} = 1.1236$$
$$1+\frac{r}{12} = (1.1236)^{1/24}$$
$$\frac{r}{12} = 0.0049$$
$$r = 0.0584$$

The annual interest rate at *ECU* would need to be 5.84%.

4. Now we know at *ECU*, $P = 1{,}000$, $r = 0.0475$, and $n = 6$. We need to determine t so that we have \$1,123.60 in the savings account. Therefore, we need to solve

expressions and to transform and solve equations.

Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

$$1{,}000\left(1+\frac{0.0475}{6}\right)^{6t}=1{,}123.6$$
$$(1.0079)^{6t}=1.1236$$
$$6t(\ln 1.0079)=\ln(1.1236)$$
$$t=\frac{\ln(1.1236)}{6\ln(1.0079)}$$
$$t\approx 2.47$$

It would take $2\frac{1}{2}$ years for you to have as much money at *ECU* as you would have at *FSB* at the end of 2 years.

Plan B

The function expressing the amount, *A* dollars, in an account that earns interest continuously in terms of *t* years is

$$A=Pe^{rt}$$

where r = the continuous compound interest rate.

1. If r = 0.05 and t = 2, we need to solve

$$P\cdot e^{(0.05)2}=1{,}123.6$$
$$e^{0.1}P=1{,}123.6$$
$$P=1{,}123.6\,e^{-0.1}$$
$$P=1{,}016.68$$

Our principal at *ECU* again would need to be $1,016.68 to break even with *FSB* after 2 years of saving.

2. This time, we need to solve

$$1000(e^{r\cdot 2})=1{,}123.6$$
$$e^{2r}=1.1236$$
$$2r=\ln(1.1236)$$
$$r=0.0583$$

ECU's interest rate would need to be 5.83%.

3. For *ECU*, we need to solve the equation

$$1{,}000(e^{r \cdot 10}) = 2{,}000$$
$$e^{10r} = 2$$
$$10r = \ln(2)$$
$$r = \frac{\ln(2)}{10}$$
$$r = 0.069$$

ECU would have to pay 6.9% continuous compound interest.

After 10 years at *FSB*, we would have $1{,}000(1.06)^{10} = \$1{,}790.84$ in the account.

Extension Questions:

- In Plan A, problem 1, the function rules for *FSB* and *ECU* respectively are $Y_1 = 1{,}000(1.06)^x$ and $Y_2 = 1{,}017(1.0125)^{4x}$. The variable *x* has been used to represent time on the graphing calculator. Which generates more money? How do you decide?

  This can be shown with tables. Here are the tables for these functions:

X	Y1	Y2
0	1000	1017
1	1060	1068.8
2	1123.6	1123.3
3	1191	1180.5
4	1262.5	1240.6
5	1338.2	1303.8
6	1418.5	1370.3

Y1=1000(1.06)^X

For x less than 2, the y-values are more for Y_2. That is, during the first 2 years there is more money in the account at ECU. The amounts are almost equal at x equals 2. They break even at 2 years. For x-values greater than 2, Y_2 is greater. After that, FSB is the better place to have your money.

The graph also shows this solution.

```
WINDOW
 Xmin=1.95
 Xmax=1.975
 Xscl=0
 Ymin=1120
 Ymax=1122
 Yscl=0
 Xres=■
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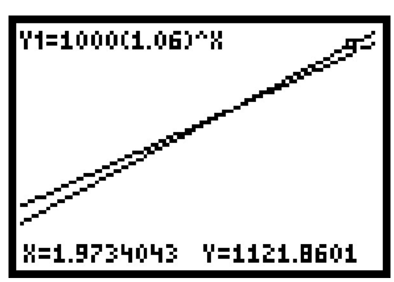


Another way to see this is to solve the equation to find when the two functions have the same y-value.

$$1,000(1.06)^{x} = 1,017(1.0125)^{4x}$$
$$\log\left[1,000(1.06)^{x}\right] = \log\left[1,017(1.0125)^{4x}\right]$$
$$\log 1,000 + \log(1.06)^{x} = \log 1,017 + \log(1.0125)^{4x}$$
$$\log 1,000 + x\log(1.06) = \log 1017 + 4x\log(1.0125)$$
$$x\log(1.06) - 4x\log(1.0125) = \log 1,017 - \log 1,000$$
$$x\left[\log(1.06) - 4\log(1.0125)\right] = \log 1,017 - \log 1,000$$
$$x = \frac{\log 1,017 - \log 1,000}{\left[\log(1.06) - 4\log(1.0125)\right]} = 1.965$$

In about 1.965 years the functions have the same amount.

- What kinds of equations did you have to solve in Plan A?

  Problem 1 had a linear equation.

  $$P\left(1+\frac{0.05}{4}\right)^{8} = 1.123.60 \text{ for } P$$

  Problem 2 had an exponential equation.

  $$1,000\left(1+\frac{r}{12}\right)^{24} = 1,123.60 \text{ for } r$$

  Problem 3 had exponential and required logarithms.

  $$1,000\left(1+\frac{0.475}{6}\right)^{6t} = 1,123.60 \text{ for } t$$

- Suppose you are given the value of r_s in the continuous compound interest situation, $Y = A(1+r_s)^{x}$. How can you determine possible values of r_c in the compound interest situation that make continuous compound interest the better deal?

We want the continuously compounded interest amount earned to be greater than the compound interest amount.

$$Ae^{r_c x} > A(1+r_s)^x$$
$$e^{r_c x} > (1+r_s)^x$$
$$e^{r_c} > (1+r_s)$$
$$r_c > \ln(1+r_s)$$

Choose r_c *so that* $r_c > \ln(1+r_s)$.

A Graduation Present

Suppose your grandparents offer you $3,500 as a graduation gift. However, you will receive the gift only if you agree to invest the money for at least 4 years. At that time, you hope to purchase a new car as a college graduation present to yourself and hope to make a downpayment of $5,000.

1. At what interest rate, compounded monthly, would you need to invest your money so that you have at least $5,000 accumulated in 4 years?

2. If you invest your money at a 5% interest rate compounded daily, how long would it take you to accumulate $5,000?

3. You decide to ask your grandparents for more money so you can plan to make a larger downpayment. You plan to save this money at a credit union that offers 6% interest compounded continuously. How much should you ask for if you want at least $5,000 in the account in 4 years?

   You know your grandfather will see the need for more money more clearly if you present the information in a table or graph. However, your grandmother, a retired math teacher, will expect an algebraic explanation.

Notes

Materials:

Graphing calculator

Note: Prerequisite experiences for this problem are that students will have been exposed to the interest formulas.

Compound interest:

$A = P\left(1+\frac{r}{n}\right)^{nt}$ where n is the number of times compounded each year, P is the amount invested, A is the amount in the account after t years, and r is the annual interest rate.

Continuous interest:

$A = P(e^{rt})$ where P is the amount invested, A is the amount in the account after t years, and r is the annual interest rate.

Scaffolding Questions:

- If interest is computed n times per year, what would be the function rule expressing the amount of money (A dollars) in a savings account as a function of the number of years (t) the money is in the account?
- What function rule do you use if interest is computed continuously?
- As you solve for different parameters in these functions, what types of equations do you encounter?
- What methods do you have for solving exponential equations?
- What methods seem to work best in these situations?

Sample Solutions:

1. The function expressing the amount of money, A dollars, in an account in terms of years of deposit, t years, is

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

where P is your initial deposit, r is the annual interest rate, and n is the number of compounding periods per year.

Since P = 3,500, n = 12, t = 4, and A = 5,000 we have

$$3{,}500\left(1+\frac{r}{12}\right)^{12*4} \geq 5{,}000$$

$$\left(1+\frac{r}{12}\right)^{48} \geq \frac{5{,}000}{3{,}500}$$

$$1+\frac{r}{12} \geq \left(\frac{10}{7}\right)^{\left(\frac{1}{48}\right)}$$

$$= 1.007458$$

$$\frac{r}{12} \geq 0.007458$$

$$r \geq 0.0895$$

Therefore, we would need to find a savings program that paid at least 8.95% interest compounded monthly.

2. In this case, we know $P = 3{,}500$, $r = 0.05$, $n = 365$, and $A = 5{,}000$. We must solve for t as follows:

$$3{,}500\left(1+\frac{0.05}{365}\right)^{365t} = 5{,}000$$
$$(1.000136986)^{365t} = \frac{10}{7}$$

Take the natural log of both sides and apply the power and quotient properties of logarithms to get

$$365t(\ln(1.000136986)) = \ln(10) - \ln(7)$$
$$365t = \frac{0.3566749}{0.0001369766}$$
$$= 2{,}603.905424$$
$$t = 7.134$$

Therefore, it would require more than seven years to accumulate \$5,000.

3. Now we need to determine how much to deposit initially if interest is compounded continuously, $r = 0.06$, $t = 4$, $A = 5{,}000$.

Use the continuous compound interest function

$$P(e^{0.06*4}) = 5{,}000$$
$$P(1.1271249) = 5{,}000$$
$$P = 3{,}933.139$$

We would need \$3,933.14 as our initial deposit.

$3,933.14 – $3,500 = $433.14

Algebra II TEKS Focus: (2A.11) Exponential and logarithmic functions. The student formulates equations and inequalities based on exponential and logarithmic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

(D) determine solutions of exponential and logarithmic equations using graphs, tables, and algebraic methods.

(E) determine solutions of exponential and logarithmic inequalities using graphs and tables.

(F) analyze a situation modeled by an exponential function, formulate an equation or inequality, and solve the problem.

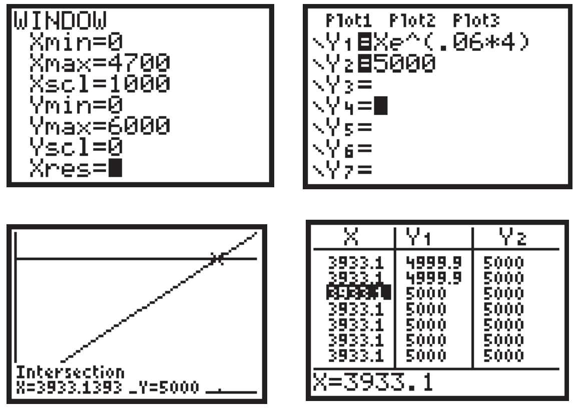
Notes

Additional Algebra II TEKS: (2A.2) Foundations for functions. The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

The student is expected to:

(A) use tools including factoring and properties of exponents to simplify expressions and to transform and solve equations.

The following graph and table show the same solution:



Therefore, you should ask your grandparents to increase their gift by at least $433.14.

Extension Questions:

- Consider the situation in problem 1. Suppose you invest twice as much money for the same amount of time at the same interest rate, 8.95%. Will the amount of money earned be more than, less than, or equal to twice as much as the amount earned for $3,500?

  The amount earned will be twice as much.

  The amount after 4 years is

  $$3,500\left(1+\frac{.0895}{12}\right)^{12(4)}=5,009.918667\approx 5,009.92.$$

  The amount earned is $5,009.92 – $3,500 = $1,509.92.

  If twice as much, $7,000, is invested, the amount earned is

  $$7,000\left(1+\frac{.0895}{12}\right)^{12(4)}=10,019.83733\approx 10,019.84$$

  The amount earned is $10,019.84 – $7,000 = 3,019.84 = 2($1,509.92).

- Show that this is true for any amount invested at a rate, *r*, compounded *n* times per year for *t* years.

  The original amount invested is represented by P. The amount earned is

  $$A = P\left(1+\frac{r}{n}\right)^{nt}$$

  The interest earned is the amount earned minus the amount invested.

  $$P\left(1+\frac{r}{n}\right)^{nt} - P$$

  If the amount is multiplied by 2, it is 2P. The amount earned is

  $$2P\left(1+\frac{r}{n}\right)^{nt}$$

  The interest earned is the amount earned minus the amount invested.

  $$2P\left(1+\frac{r}{n}\right)^{nt} - 2P = 2\left\{P\left(1+\frac{r}{n}\right)^{nt} - P\right\}$$

  If the amount invested is multiplied by two, then the amount earned is always twice the original amount earned.

The student work on the next page demonstrates use of algebraic process and skills, but does not communicate the solution strategy.

Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

A Graduation Present

$Y = A(1 + R/12)^{X \cdot 12}$ (monthly)

R = interest rate

X = # of times compounded per year

① * $5000 = 3500\ (1 + \frac{R}{12}) \wedge (4)(12)$
(Y)↑ ↑(A) ↑(X)

* $\frac{5000}{3500} = \frac{3500\ (1 + \frac{R}{12})^{(48)}}{3500}$

* $1.428^{(1/48)} = (1 + \frac{R}{12})^{\frac{48}{1} * (1/48)}$

* $1.0074 = 1 + \frac{R}{12}$
$-1 \qquad -1$

* $12 \times .0074 = R/12 \times 12$

$.0894 = R$

$\boxed{8.94\ \% = R}$ This is the interest rate in which he would need to make \$5000! in 4yr.

② * $\frac{5000}{3500} = \frac{3500\ (1 + \frac{.05}{365})}{3500} \wedge (X)(365)$ (Daily)

* $1.428 = (1 + .0001369)^{365X}$

$1.428 = (1.0001369)^{365X}$

* $\frac{\log 1.428}{\log 1.0001369} = \frac{365 \times \log 1.0001369}{\log 1.0001369}$

* $\frac{2602.624}{365} = \frac{365X}{365}$

$\boxed{* \ 7.1304 \text{ yrs} = X}$ This is how long it will take for him to make \$5000!

③ $y = Pe^{rt}$

* $5000 = Pe^{.06(4)}$

* $5000 = Pe^{.24}$

* $\frac{5000}{1.271} = \frac{P(1.271)}{1.271}$

$\boxed{3933.14 = P}$

$$\begin{array}{r} 3933.14 \\ -\ 3500.00 \\ \hline \$\ 433.13 \end{array}$$

That's how much extra we need to ask for.

Chapter 7:
Rational Functions

Paintings on a Wall

In order to optimize the viewing space for its patrons, a museum has placed size restrictions on rectangular paintings that will be hung on a particular wall.

The perimeter of a painting must be between 64 inches and 100 inches, inclusive. The area of the painting must be between 200 square inches and 500 square inches.

1. You are in charge of determining possible perimeter and area combinations for paintings to be hung on the wall. Write inequalities to describe the perimeter and area restrictions in terms of the length and the width of the rectangles. Graph the resulting system.

2. Algebraically and with technology, determine the vertices of the region defined by the inequalities in problem 1.

3. Describe the location of the points on your graph where the dimensions of the painting result in each of the following:

   a. The perimeter and area are acceptable.
   b. The perimeter is too short or too long, but the area is acceptable.
   c. The perimeter is acceptable, but the area is too small or too large.
   d. Neither the perimeter nor the area is acceptable.

   Explain how you arrived at your responses.

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus: (2A.3) Foundations for functions. The student formulates systems of equations and inequalities from problem situations, uses a variety of methods to solve them, and analyzes the solutions in terms of the situations.

The student is expected to:

(A) analyze situations and formulate systems of equations in two or more unknowns or inequalities in two unknowns to solve problems.

(B) use algebraic methods, graphs, tables, or matrices, to solve systems of equations or inequalities.

(C) interpret and determine the reasonableness of solutions to systems of equations or inequalities for given contexts.

Additional Algebra II TEKS: (2A.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

Scaffolding Questions:

- What information is known?
- What are you expected to show?
- What compound inequality can you write to describe the restrictions on the perimeter?
- What compound inequality can you write to describe the restrictions on the area?
- How can you use the inequalities to write functions that describe the boundaries of the region containing acceptable values for width and length of a painting?
- Describe the process you might use to graph the compound inequality.
- What functions must be graphed to model a combined inequality such as $2 \le x + y \le 5$?
- What kind of systems must you solve to get the vertices of the boundary?
- What algebraic method will you use to solve each system? Why?
- Point to each region in the plane determined by the boundary functions and describe whether the perimeter and area meet the requirements in that region.

Sample Solutions:

1. Let *l* = the length in inches of a painting and *w* = the width in inches of the painting.

   Since the perimeter of the painting must be between 64 inches and 100 inches, we know that

   $$64 \le 2(l + w) \le 100$$

   which gives

   $$32 \le l + w \le 50$$
   $$32 - w \le l \le 50 - w$$

Since the area of the painting must be between 200 square inches and 500 square inches, we have

$$200 \le lw \le 500$$

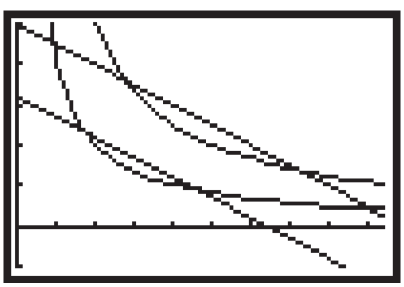
which gives

$$\frac{200}{w} \le l \le \frac{500}{w}$$

On the graphing calculator, the length will be represented by *Y* and the width will be represented by *x*.

Function Rule:	Calculator Rule:
$l = 32 - w$	$Y_1 = 32 - x$
$l = 50 - w$	$Y_2 = 50 - x$
$l = \frac{200}{w}$	$Y_3 = \frac{200}{x}$
$l = \frac{500}{w}$	$Y_4 = \frac{500}{x}$

Here is the calculator graph of the system with window settings $0 \le x \le 47$, $-10 \le y \le 50$:



The region that is the solution set to the system of inequalities is the closed region between the two lines and the two curves.

2. We use the intersect feature of the calculator to determine the coordinates of the vertices. Starting with the upper left vertex and moving counterclockwise around the region, they are:

The intersection of Y_2 and Y_3: A(4.38, 45.62)
The intersection of Y_1 and Y_3: B(8.52, 23.48)

The student is expected to:

(A) identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations.

(B) collect and organize data, make and interpret scatterplots, fit the graph of a function to the data, interpret the results, and proceed to model, predict, and make decisions and critical judgments.

Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 4: The student will formulate and use linear equations and inequalities.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

The intersection of Y_1 and Y_3: C(23.48, 8.52)
The intersection of Y_2 and Y_3: D(45.62, 4.38)
The intersection of Y_2 and Y_4: E(36.18, 13.82)
The intersection of Y_2 and Y_4: F(13.82, 36.18)

To find the vertices algebraically, we solve the following systems by substitution:

$$Y_1 = 32 - x$$
$$Y_3 = \frac{200}{x}$$
$$\text{Let } Y_1 = Y_3$$
$$32 - x = \frac{200}{x}$$
$$32x - x^2 = 200$$
$$x^2 - 32x + 200 = 0$$
$$x = \frac{32 \pm \sqrt{(-32)^2 - 4 \cdot 1 \cdot 200}}{2}$$
$$x = 23.48 \text{ or } x = 8.52$$

This gives the *x*-coordinate for vertices B and C.

Substitute for *x* in Y_1 to get the *y*-coordinates.

$$Y_2 = 50 - x$$
$$Y_3 = \frac{200}{x}$$
$$\text{Let } Y_2 = Y_3$$
$$50 - x = \frac{200}{x}$$
$$50x - x^2 = 200$$
$$x^2 - 50x + 200 = 0$$
$$x = \frac{50 \pm \sqrt{(-50)^2 - 4 \cdot 1 \cdot 200}}{2}$$
$$x = 45.62 \text{ or } x = 4.38$$

This gives the *x*-coordinate for vertices D and A.

Substitute for *x* in Y_2 to get the *y*-coordinates.

$$Y_2 = 50 - x$$

$$Y_4 = \frac{500}{x}$$

Let $Y_2 = Y_4$

$$50 - x = \frac{500}{x}$$

$$50x - x^2 = 500$$

$$x^2 - 50x + 500 = 0$$

$$x = \frac{50 \pm \sqrt{(-50)^2 - 4 \cdot 1 \cdot 500}}{2}$$

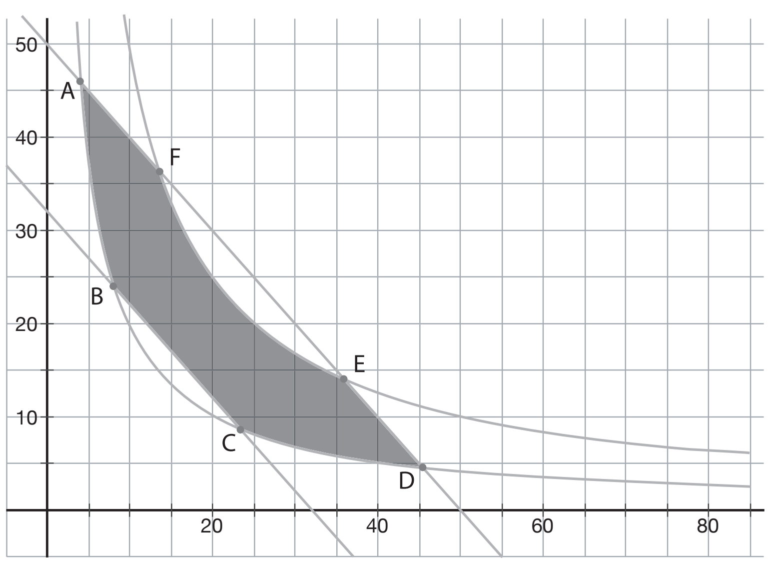
$$x = 36.18 \text{ or } x = 13.82$$

This gives the *x*-coordinate for vertices E and F.

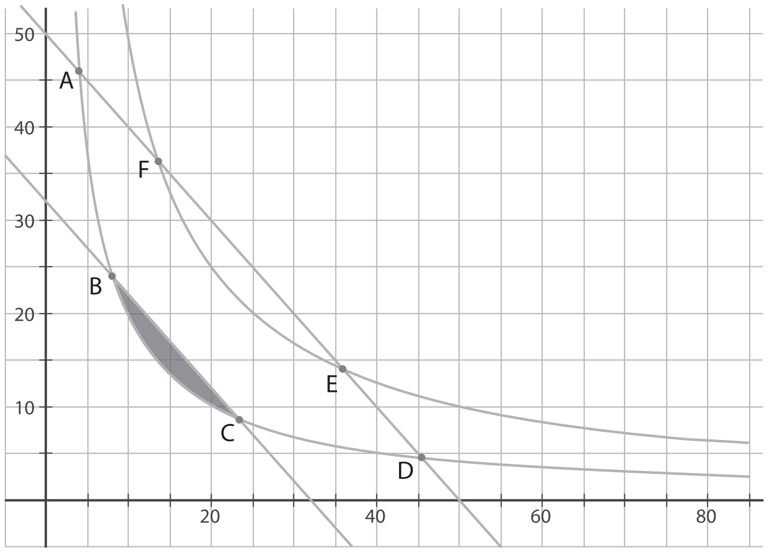
Substitute for *x* in Y_2 to get the *y*-coordinates.

3. Remember that the coordinates *(x, y)* of the points in the plane represent (width, length) of the painting. Therefore, we can consider points only in the first quadrant.

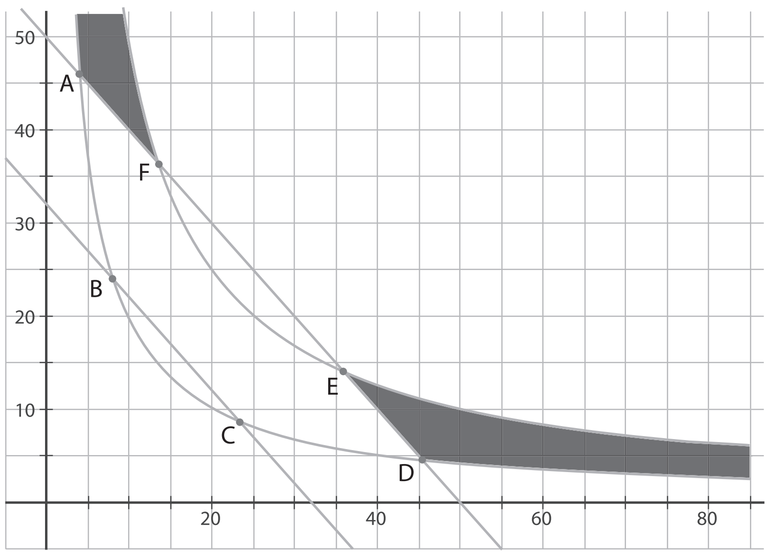
   a. The region where the perimeter and area are acceptable is the closed region between the lines and the curves.



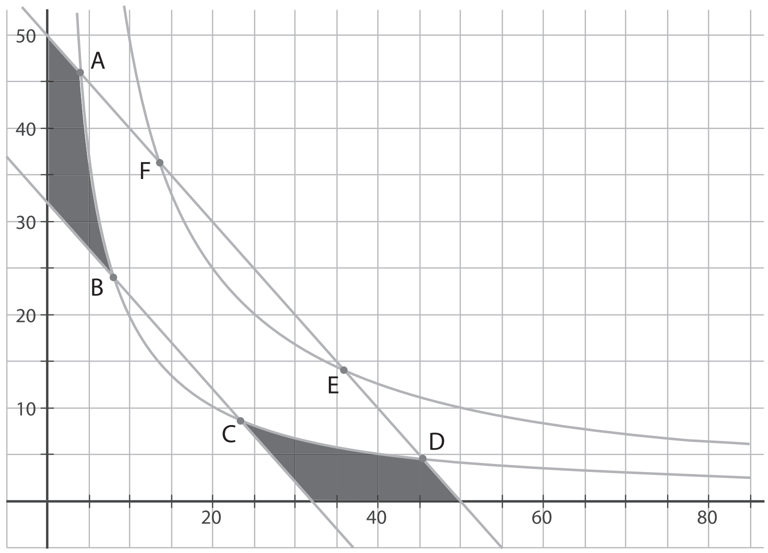
   b. For the perimeter to be too short and the area acceptable, we need first quadrant points below the line $y = 32 - x$ but between or on the curves.



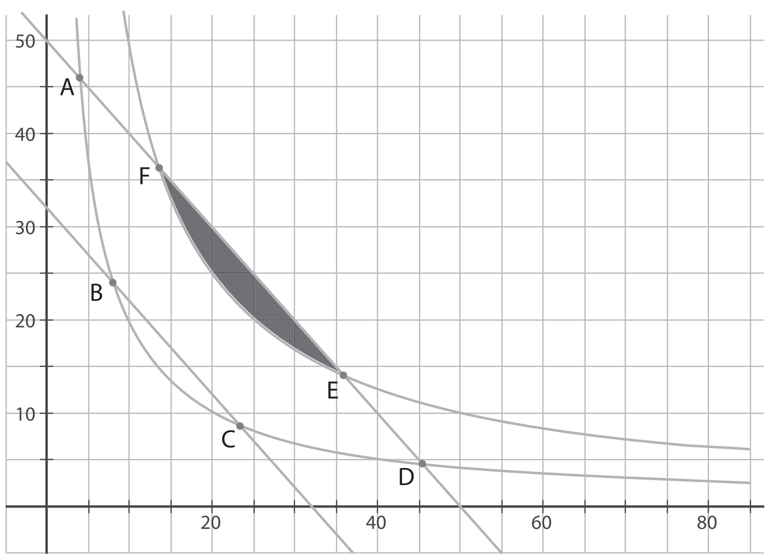
For the the perimeter to be too long and the area acceptable, we need first quadrant points above the line $y = 50 - x$ but between or on the curves.



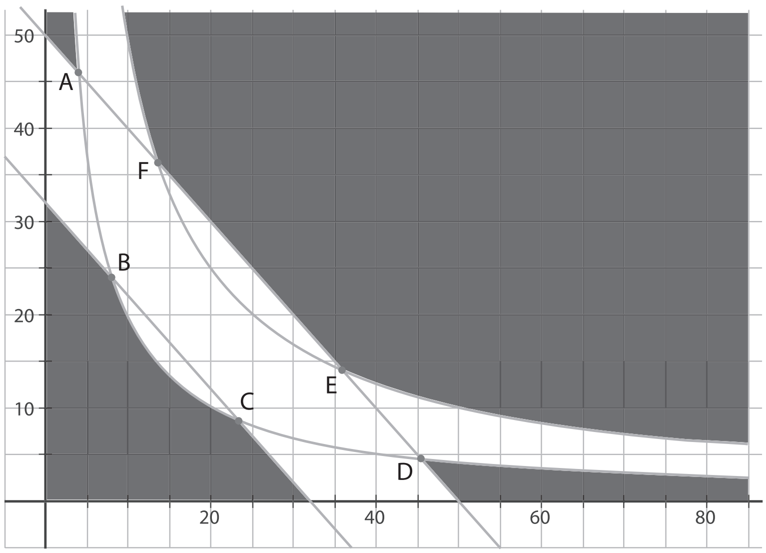
c. For the perimeter to be acceptable and the area too small, we need first quadrant points between or on the lines but below the curve $y = \dfrac{200}{x}$.



For the perimeter to be acceptable and the area too large, we need first quadrant points between or on the lines but above the curve $y = \dfrac{500}{x}$.



d. If the perimeter and area are both unacceptable, we need first quadrant points both outside the lines and the curves.



Extension Questions:

- How is the region representing acceptable dimensions for a painting different from regions you encountered in linear programming problems?

  The boundaries are not all defined by linear functions. Three boundaries are segments, defined by the two linear (perimeter) functions. Three boundaries are defined by the two reciprocal (area) functions.

- Which vertex or vertices minimize perimeter and area, subject to the restrictions? How do you know?

  To minimize both perimeter and area you must be at the intersection of $Y_1 = 32 - x$ and $Y_3 = \frac{200}{x}$ *because these give the lower limits on perimeter and area. This is vertex B or C. The painting can have dimensions 8.52 inches by 23.48 inches. To maximize both perimeter and area, you must be at the intersection of* $Y_2 = 50 - x$ and $Y_4 = \frac{500}{x}$ *because these give the upper limits on perimeter and area. This is vertex E or F. The painting can have dimensions 36.18 inches by 13.82 inches.*

- Does the graph have any symmetry?

  Yes. It is symmetric with respect to the line y = x.

- Suppose the area restriction were replaced with the restriction that the diagonal of a painting must be between 25 inches and 40 inches in length. How will this change your responses to the previous two questions?

  The restrictions on the perimeter stay the same, but we must replace area restrictions with diagonal restrictions.

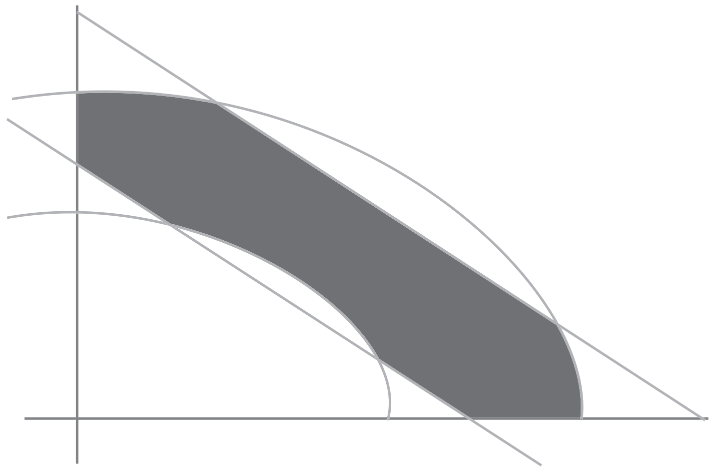
Let d = the length of the painting's diagonal in inches. Then, since $d^2 = l^2 + w^2$,

$$25 \le d \le 40 \Rightarrow 25^2 \le d^2 \le 40^2$$
$$25^2 \le l^2 + w^2 \le 40^2$$
$$25^2 - w^2 \le l^2 \le 40^2 - w^2$$
$$\sqrt{25^2 - w^2} \le l \le \sqrt{40^2 - w^2}$$

The area boundary equations are replaced with

$$l = \sqrt{25^2 - w^2} \text{ and } l = \sqrt{40^2 - w^2}$$

The graph of the region representing acceptable dimensions is the first quadrant region between the lines and between the semi-circles. It does not include points on the axes, since that would give you zero width or zero length.



To find where the perimeter lines and diagonal circles intersect, we solve the systems consisting of Y_1 *and* Y_3 *and of* Y_2 *and* Y_4*. For example,*

$$Y_1 = 32 - x$$
$$Y_3 = \sqrt{25^2 - x^2}$$
$$\text{Let } Y_1 = Y_3$$
$$32 - x = \sqrt{25^2 - x^2}$$
$$x^2 - 64x + 32^2 = 25^2 - x^2$$
$$2x^2 - 64x + 32^2 - 25^2 = 0$$

Apply the quadratic formula to get x = 8.48 inches or x = 23.52 inches.

Similarly, we solve for the intersection of Y_2 and Y_4 *to get x = 11.77 inches or x = 38.23 inches.*

The vertex points are (8.48 , 23.52), (23.52, 8.48), (11.77, 38.23) and (38.23, 11.77).

Saline Solution

You have been hired as an intern at the Sodium Solutions factory over the summer to earn money for college. Your job requires you to dilute a salt and water solution that is required for various applications at the factory. A bottle of solution contains 1 liter of a 20% salt solution. This means that the concentration of salt is 20% of the entire solution.

1. The supervisor has asked you to dilute the solution by adding water to the bottle in half-liter amounts and to record the amount of water in the bottle after each addition of water, as well as the new concentration of salt.
2. Find a function that models the concentration of salt in the whole solution as you add water. Explain how you determined your function.
3. Describe the graph of the function.
4. Name the parent function for the family of functions to which this graph belongs.
5. Describe how the graph of the function related to the graph of the parent function.
6. How much water should you add to get a 2.5% salt solution?

Notes

Materials:
Graphing calculator

Algebra II TEKS Focus:
(2A.10) Rational functions. The student formulates equations and inequalities based on rational functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

(D) determine the solutions of rational equations using graphs, tables, and algebraic methods.

(F) analyze a situation modeled by a rational function, formulate an equation or inequality composed of a linear or quadratic function, and solve the problem.

Additional Algebra II TEKS:
(2A.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

The student is expected to:

(B) collect and organize data, make and interpret scatterplots, fit the graph of a function to the data, interpret the results, and proceed to model, predict, and make decisions and critical judgments.

(2A.4) Algebra and geometry. The student connects algebraic and geometric representations of functions.

Scaffolding Questions:

- How could you write a ratio to show the amount of salt in liters to the total amount of the solution in liters?
- How much water is added each time?
- How much salt is in the entire solution each time water is added?
- How much of a 1-liter bottle solution is salt?
- How can you use the ratio of salt to total solution to determine the new concentration of salt with each addition of water?

Sample Solutions:

1. The 1-liter bottle of solution originally contained 20% salt, or 0.2 liter. The ratio of salt to water is 20 : 100. As water is added, the amount of salt remains the same, but the amount of the whole solution increases by half (0.5) a liter. Each time water is added, the concentration of salt is reduced. This is calculated by finding the decimal equivalent of the ratio of salt to whole solution.

   The following table shows the results of the dilution process:

Salt (L)	Water added (L)	Whole solution (L)	Salt concentration (L)
0.2	0	1.0	0.20
0.2	0.5	1.5	0.133
0.2	1.0	2.0	0.10
0.2	1.5	2.5	0.08
0.2	2.0	3.0	0.067
0.2	2.5	3.5	0.057
0.2	3.0	4.0	0.05
0.2	3.5	4.5	0.044
0.2	4.0	5.0	0.04

2. The amount of the whole solution increases while the concentration of salt decreases. However, the amount of salt remains constant throughout the process. The function can be modeled by the following:

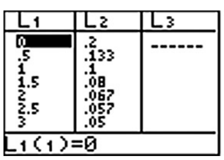
$$\text{Concentration of salt} = \frac{\text{amount of salt}}{\text{amount of whole solution}}$$

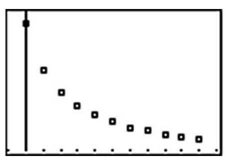
Let x = the amount of water added. Let y = the concentration of salt.

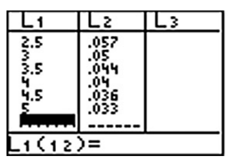
$$y = \frac{0.2}{x+1}$$

The whole solution represents the original 1 liter plus the half-liters that were added.

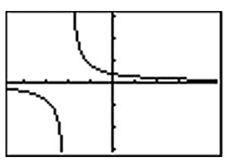
3. Entering the values for the amount of water added x into L_1, and the salt concentration y into L_2 of the graphing calculator produces the following table and stat plot.







Entering the function into the graphing calculator produces a transformation of the graph of the function $y = \frac{1}{x}$. The portion of the graph that represents this problem situation is the portion of the graph in the first quadrant.



4. Specifically, $y = \frac{0.2}{x+1}$ is a transformation of the parent function $y = \frac{1}{x}$.

5. The function has been translated to the left one unit and compressed vertically.

The student is expected to:

(A) identify and sketch graphs of parent functions, including linear ($f(x) = x$), quadratic ($f(x) = x^2$), exponential ($f(x) = a^x$), and logarithmic ($f(x) = \log_a x$) functions, absolute value of x ($f(x) = |x|$), square root of x ($f(x) = \sqrt{x}$), and reciprocal of x ($f(x) = 1/x$).

(B) extend parent functions with parameters such as a in $f(x) = a/x$ and describe parameter changes on the graph of parent functions.

Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 9: The student will demonstrate an understanding of percents, proportional relationships, probability, and statistics in application problems.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

6. Using the table feature on the calculator shows that 7 liters of water must be added to obtain a 2.5% (0.025) concentration of salt.

X	Y1	
7	.025	
8	.02222	
9	.02	
10	.01818	
11	.01667	
12	.01538	
13	.01429	

X=7

This problem may also be answered by solving the equation $0.025 = \dfrac{0.2}{x+1}$.

$$0.025(x+1) = 0.2$$
$$0.025x + 0.025 = 0.2$$
$$0.025x = 0.175$$
$$x = 7$$

Extension Questions:

- What are the domain and range values for the mathematical function $y = \dfrac{0.2}{x+1}$?

  The denominator of the fraction may not be 0. Thus, the domain includes all real numbers except -1.

  The range includes all reals except 0, because $\dfrac{0.2}{x+1}$ *will never be equal to zero. It can also be seen from the graph that the y-value is never zero.*

- What are the domain and range values for the problem situation?

  Since x represents an amount of water added, it could be any number greater than or equal to zero: $x \geq 0$.

  The y-value represents a concentration of salt. The greatest value is when x = 25% or 0.25. As water is added the concentration of salt decreases but will not reach zero.

$$0 < y \leq 0.25$$

- If the function you wrote to model a similar situation had been $y = \dfrac{0.3}{x+1}$, what would you know about the original solution?

The original solution would have contained 1 liter but it would have been 30% salt instead of 20% salt.

- Suppose your supervisor asked you to begin with a bottle of solution that contains 2 liters of a 25% salt solution and to follow the same procedure to dilute the solution. Describe how to determine when the concentration of salt will be 2.5%.

  The change in the function rule is that 1 liter becomes 2 liters and 0.2 becomes 0.25.

  The new function rule that models this situation is $y = \frac{0.25}{x+2}$ *, where y represents the concentration and x represents the amount added to the 2-liter solution.*

  To determine when the solution is 2.5% salt, let y = 0.025.

$$\begin{aligned} 0.025 &= \frac{0.25}{x+2} \\ 0.025(x+2) &= 0.25 \\ 0.025x + 0.05 &= 0.25 \\ 0.025x &= 0.2 \\ x &= 8 \end{aligned}$$

Eight liters of water would have to be added to the solution.

- How much 2.5% solution is needed to dilute the original 1 liter of 20% solution to give a 10% solution?

  Let x represent the amount of a 2.5% solution. 0.025x represents the amount of salt in that solution. 1 + x represents the total amount. The function for the concentration of salt in this new solution is

$$\begin{aligned} 0.1 &= \frac{0.2 + 0.025x}{x+1} \\ 0.1(x+1) &= 0.2 + 0.025x \\ 0.1x + 0.1 &= 0.2 + 0.025x \\ 0.075x &= 0.1 \\ x &= 1\frac{1}{3} \end{aligned}$$

Thus, $1\frac{1}{3}$ *liters of the new solution are needed to make a 10% concentration.*

Pizza Wars, Part 2

Little Nero's	
Giant	$15.99
Extra Large	$12.99
Large	$10.99

Donatello's	
Large	$15.88
Medium	$12.88
Small	$10.88

In a recent advertisement, the pizza restaurant Little Nero's made each of the following claims comparing the size of its pizzas to the size of comparably priced pizzas at Donatello's, Little Nero's main competitor:

- Little Nero's giant pizza is 65% bigger than Donatello's large pizza.
- Little Nero's extra large pizza is 77% bigger than Donatello's medium pizza.
- Little Nero's large pizza is 96% bigger than Donatello's small pizza.

By claiming, for example, that its large pizza is "96% bigger" than Donatello's small pizza, Little Nero's is asserting that the area of its large pizza is 96% bigger than the area of Donatello's small pizza.

1. The diameter of Little Nero's large pizza is 14 inches and the diameter of Donatello's small pizza is 10 inches. Determine whether the claim "Little Nero's large pizza is 96% bigger than Donatello's small pizza" is valid.

2. a) The diameter of Donatello's medium pizza is 12 inches, and Little Nero's extra large pizza is, in fact, 77% bigger than Donatello's medium pizza. Find, to the nearest inch, the diameter of Little Nero's extra large pizza. How much longer than the diameter of Donatello's medium pizza is the diameter of Little Nero's extra large pizza? How much longer is the radius?

   b) The diameter of Little Nero's giant pizza is 18 inches, and Little Nero's giant pizza is, in fact, 65% bigger than Donatello's large pizza. Find, to the nearest inch, the diameter of Donatello's large pizza. What can be said about the corresponding diameters for all three pairs of comparably priced pizzas at Little Nero's and Donatello's? What can be said about the corresponding radii?

3. Assume that, for any pizza at Donatello's, the radius of a comparably priced pizza at Little Nero's is 2 inches longer.

   a) If *r* represents the radius of a pizza at Donatello's, find an algebraic expression that gives the exact value (in terms of π if needed) for each of the following:

      i) the area of a comparably priced pizza at Little Nero's

      ii) the difference between the area of the Donatello's pizza and the area of a comparably priced Little Nero's pizza

      iii) as a percentage of the Donatello's pizza, how much bigger a comparably priced Little Nero's pizza would be

   b) Find a rule for the function *P(r)* that gives the percentage described in iii) above as a function of *r*.

   c) Give the domain and range of this function for the pizza problem context.

   d) Use technology to produce a graph of this function. Describe what the graph represents. Then verify that, for each of the three pairs of comparably priced pizzas given above, the graph indicates the correct percentage.

   e) Describe all asymptotic behavior for this function. Then explain the meaning of this behavior in the given context.

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus:
(2A.10) Rational functions. The student formulates equations and inequalities based on rational functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

(A) use quotients of polynomials to describe the graphs of rational functions, predict the effects of parameter changes, describe limitations on the domains and ranges, and examine asymptotic behavior.

(B) analyze various representations of rational functions with respect to problem situations.

(C) determine the reasonable domain and range values of rational functions, as well as interpret and determine the reasonableness of solutions to rational equations and inequalities.

(D) determine the solutions of rational equations using graphs, tables, and algebraic methods.

(F) analyze a situation modeled by a rational function, formulate an equation or inequality composed of a linear or quadratic function, and solve the problem.

Scaffolding Questions:

- When comparing two unequal quantities, how can we determine whether the larger quantity is 96% bigger than the smaller quantity?
- If the diameter (or the radius) is 77% bigger, is it true that the area is 77% bigger as well?
- What expression represents the difference in the areas of a pizza of radius r and a pizza of radius 6? What equation can we solve that will give us the appropriate radius?
- How can we express the area of a pizza whose radius is 2 more than r?
- Describe how to convert any given ratio to a percentage.
- What does r or $P(r)$ represent? What must be true about r or $P(r)$ in this context?
- What point on the graph of $P(r)$ will confirm that (for example) when the radius at Donatello's is 7 inches, a comparably priced pizza at Little Nero's (with radius 9 inches) is 65% bigger?
- As we move along the graph toward the vertical axis, what is happening to r? To $P(r)$?
- As we move along the graph away from the vertical axis, what is happening to r? To $P(r)$?
- From the graph, what can we tell about differences in pizzas of the two restaurants?

Sample Solutions:

1. For Little Nero's large pizza:
   If d = 14 inches, then r = 7 inches and $A = \pi(7)^2 = 49\pi$ square inches.

   For Donatello's small pizza:
   If d = 10 inches, then r = 5 inches and $A = \pi(5)^2 = 25\pi$ square inches.

Percentage increase =

$$100 \cdot \frac{49\pi - 25\pi}{25\pi} = 100 \cdot \frac{24\pi}{25\pi} = 100 \cdot \frac{24}{25} = 100(0.96) = 96$$

This confirms that the claim "Little Nero's large pizza is 96% bigger than Donatello's small pizza" is valid.

2. a) For Donatello's medium pizza:

If d = 12 inches, then r = 6 inches and $A = \pi(6)^2 = 36\pi$ square inches.

If r is the radius of Little Nero's extra large pizza, then

$$100 \cdot \frac{\pi r^2 - 36\pi}{36\pi} = 77$$

$$\frac{\pi r^2 - 36\pi}{36\pi} = 0.77$$

$$\pi r^2 - 36\pi = 0.77(36\pi)$$

$$\pi r^2 = 0.77(36\pi) + 36\pi$$

$$r^2 = 0.77(36) + 36$$

$$r^2 = 1.77(36)$$

$$r = \pm\sqrt{1.77(36)}$$

$$r \approx 8 \text{ inches}$$

So, to the nearest inch, the diameter of Little Nero's extra large pizza is 16 inches, which is 4 inches longer than the diameter of Donatello's medium pizza. The radius is 2 inches longer.

Additional Algebra II TEKS: (2A.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

The student is expected to:

(A) identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations.

Connection to TAKS:

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Objective 9: The student will demonstrate an understanding of percents, proportional relationships, probability, and statistics in application problems.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

b) For Little Nero's giant pizza:
If d = 18 inches, then r = 9 inches and $A = \pi(9)^2 = 81\pi$ square inches.

If r is the radius of Donatello's large pizza, then

$$
\begin{aligned}
100 \cdot \frac{81\pi - \pi r^2}{\pi r^2} &= 65 \\
\frac{81\pi - \pi r^2}{\pi r^2} &= 0.65 \\
81\pi - \pi r^2 &= 0.65\pi r^2 \\
81\pi &= 0.65\pi r^2 + \pi r^2 \\
1.65\pi r^2 &= 81\pi \\
r^2 &= \frac{81}{1.65} \\
r &= \pm\sqrt{\frac{81}{1.65}} \\
r &\approx 7 \text{ inches}
\end{aligned}
$$

So, to the nearest inch, the diameter of Donatello's large pizza is 14 inches. For all three pairs of comparably priced pizzas, the diameter is 4 inches longer and the radius is 2 inches longer at Little Nero's.

3. a) i) $\pi(r+2)^2 = \pi(r^2+4r+4) = \pi r^2 + 4\pi r + 4\pi$

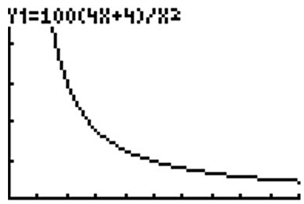
ii) $\pi(r+2)^2 - \pi r^2 = \pi r^2 + 4\pi r + 4\pi - \pi r^2 = 4\pi r + 4\pi = 4\pi(r+1)$

iii) $100 \cdot \dfrac{\pi(r+2)^2 - \pi r^2}{\pi r^2} = 100 \cdot \dfrac{4\pi r + 4\pi}{\pi r^2} = 100 \cdot \dfrac{4r+4}{r^2}$ or $\dfrac{400(r+1)}{r^2}$

b) $P(r) = 100 \cdot \dfrac{4r+4}{r^2}$ or $P(r) = \dfrac{400(r+1)}{r^2}$

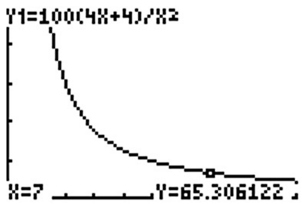
c) Domain: $0 < r \leq R$, where R is a constant representing the largest possible radius of a pizza that could be made at Donatello's (perhaps as constrained by the size of the pizza oven)

Range: $0 < P(r) \leq P_{max}$, where P_{max} is a constant representing the largest percentage difference in comparably priced pizzas that could be made at Donatello's (with radius r) and Little Nero's (with radius r + 2)

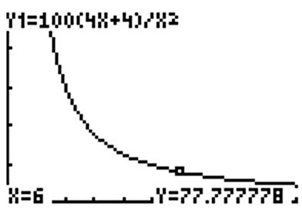
d) 


```
WINDOW
 Xmin=0
 Xmax=10
 Xscl=1
 Ymin=0
 Ymax=500
 Yscl=100
 Xres=1
```

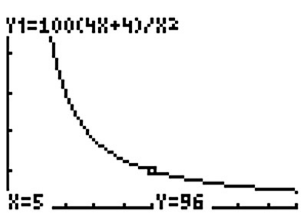
The graph above shows the percentage difference, *Y1*, comparing the size of a pizza at Little Nero's with the size of a comparably priced pizza at Donatello's, where *X* is the radius of the Donatello's pizza in inches. Although many choices for the window would be appropriate, the window used for the graph above is given at right of the graph.



The graph at left shows that the point (7, 65.3) is close to the graph of this function, thus confirming that when the radius of a Donatello's pizza is 7 inches (or the diameter is 14 inches), a comparably priced pizza at Little Nero's (with radius 9 inches, diameter 18 inches) is 65% bigger (to the nearest whole percent).



The graph at left shows that the point (6, 77.8) is close to the graph of this function, thus confirming that when the radius of a Donatello's pizza is 6 inches (or the diameter is 12 inches), a comparably priced pizza at Little Nero's (with radius 8 inches, diameter 16 inches) is 77% bigger. (Notice that Little Nero's did not round up; otherwise, the claim "78% bigger" would have been false.)



The graph at left shows that the point (5, 96) belongs to the graph of this function, thus confirming that when the radius of a Donatello's pizza is 5 inches (or the diameter is 10 inches), a comparably priced pizza at Little Nero's (with radius 7 inches, diameter 14 inches) is 96% bigger.

e) Vertical asymptote: $r = 0$

As the radius of the Donatello's pizza nears 0, the difference (as a percentage) between the Donatello's pizza and a comparably priced Little Nero's pizza (with a radius 2 inches longer) becomes large. For example, when the radius of a Donatello's pizza is 2 inches, a comparably priced pizza at Little Nero's (with radius of 4 inches) is 300% bigger.

Horizontal asymptote: P(r) = 0

As the radius of the Donatello's pizza becomes larger and larger, the difference (as a percentage) between the Donatello's pizza and a comparably priced Little Nero's pizza (with a radius 2 inches longer) nears 0%. For example, when the radius of a Donatello's pizza is 100 inches, a comparably priced pizza at Little Nero's (with radius of 102 inches) is only about 4% bigger.

Extension Questions

- Determine whether the area of a square with perimeter 12 inches is equivalent to the area of a circle with circumference (i.e., perimeter) 12 inches. If the two areas are not equivalent, determine the percentage change with respect to the area of the square.

  For a square of perimeter 12 inches, the side length, s, is $\frac{12}{4}$ *= 3 inches. Therefore, the area of the square is* 3^2 *= 9 square inches. For a circle of circumference 12 inches, the radius, r, is* $\frac{12}{2\pi} = \frac{6}{\pi}$ *inches. Therefore, the area of the circle is* $\pi(\frac{6}{\pi})^2 = \frac{36}{\pi} \approx 11.46$ *square inches. The area of the circle is about* $100 \cdot \frac{11.46 - 9}{9} \approx 27.3$ *percent bigger than the area of the square.*

- As the common perimeter changes, will the area of the circle continue to be larger than the area of the square? If so, will the area of the circle always be 27.3% larger than the area of the square?

  In order to answer the first question, we could just choose another common perimeter not equal to 12 inches and repeat the process above. This would show that, in at least one other case, the results turn out the same—that is, the area of the circle is 27.3% larger than the area of the square. However, in order to show that this is true, in general, we must show that the expression

  $$100 \cdot \frac{\textit{area of circle} - \textit{area of square}}{\textit{area of square}}$$

  is approximately 27.3 for all values of the perimeter, P. Such a general argument follows:

If P represents the common perimeter of a square and a circle, then the area of the square is given by $\left(\frac{P}{4}\right)^2 = \frac{P^2}{16}$. *The area of the circle is given by* $\pi\left(\frac{P}{2\pi}\right)^2 = \frac{P^2}{4\pi}$

Since $16 > 4\pi$, $\frac{P^2}{16} < \frac{P^2}{4\pi}$. *Therefore, the area of the circle is always larger. How much larger, as a percentage, is given by the following expression:*

$$100 \cdot \frac{\frac{P^2}{4\pi} - \frac{P^2}{16}}{\frac{P^2}{16}} = 100 \cdot \frac{P^2\left(\frac{1}{4\pi} - \frac{1}{16}\right)}{P^2\left(\frac{1}{16}\right)} = 100 \cdot \frac{\left(\frac{1}{4\pi} - \frac{1}{16}\right)}{\left(\frac{1}{16}\right)}$$

The ratio does not depend on the value of the perimeter, P. It must be a constant. Continuing to simplify,

$$100 \cdot \frac{\left(\frac{1}{4\pi} - \frac{1}{16}\right)}{\left(\frac{1}{16}\right)} = 100 \cdot \frac{16\pi\left(\frac{1}{4\pi} - \frac{1}{16}\right)}{16\pi\left(\frac{1}{16}\right)} = 100 \cdot \frac{4 - \pi}{\pi} \approx 100 \cdot .273 = 27.3$$

This proves that the area of a circle is always approximately 27.3% larger than the area of a square with the same perimeter.

Name of Problem Pizza Wars #2
Student # 2

1)

Little Nero's Large	Donatello's small
$A = \pi r^2$	$A = \pi r^2$
$A = \pi(7^2)$	$A = \pi(5^2)$
$A = 49\pi$	$A = 25\pi$
$A = 153.94$	$A = 78.54$

$78.54 \times 1.96 = 153.94$

Little Nero's large pizza is 96% larger than Donatello's small pizza.

2a) Donatello's medium

$A = \pi r^2$
$A = \pi 6^2$
$A = 113.10$
$113.10 \times 1.77 = \boxed{200.18}$

$A = \pi r^2$ (Little Nero's extra-large)
$200.18 = \pi r^2$
$63.72 = r^2$
$r = 7.98$

The diameter of Little Nero's extra large pizza is four inches longer than Donatello's medium, and the radius is 2 inches longer.

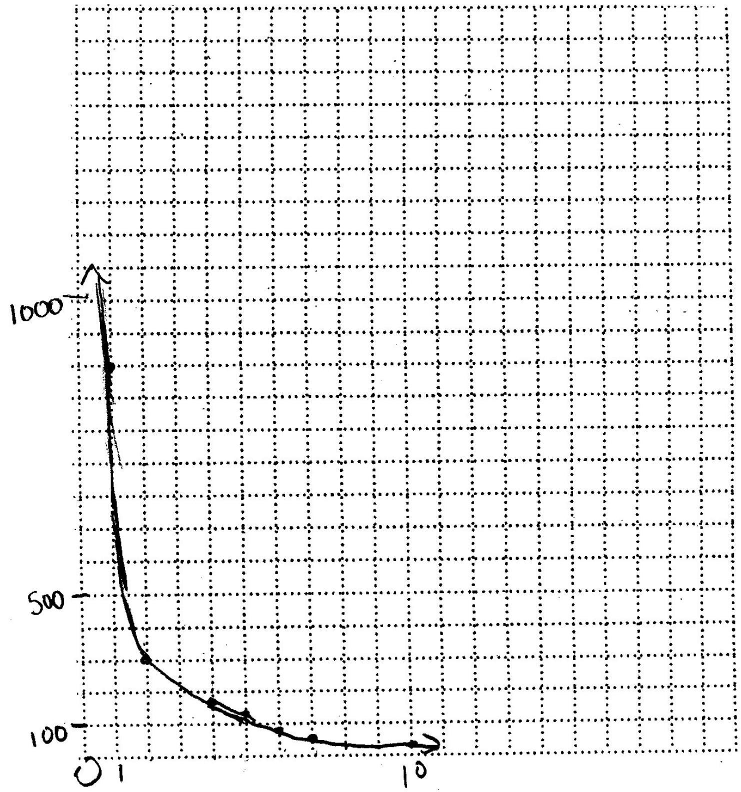
b) Little Nero's giant

$A = \pi r^2$
$A = \pi 9^2$
$A = 254.47$
$254.47 / 1.65 = 154.22$

Donatello's large
$A = \pi r^2$
$154.22 = \pi r^2$
$49.09 = r^2$
$r = 7.01$

Donatello's large pizza has a diameter of 14 inches.

For all three pairs of comparably-priced pizzas, the radii of Little Nero's are 2 inches longer than Donatello's and the diameters are 4 inches longer.



3. The graph of

$$p = \frac{400}{R} + \frac{400}{R^2}$$

3a) i) $L.N. = \pi(r+2)^2$ $D = \pi r^2$

$L.N. = \pi(r^2+4r+4)$

$L.N. = \pi r^2 + 4\pi r + 4\pi$

ii) Difference $= (\pi r^2 + 4\pi r + 4\pi) - \pi r^2$

$= 4\pi r + 4\pi$

iii) expression representing the percentage that L.N. pizza is larger, in terms of D.'s pizza

$$\left(\left(\frac{\pi r^2 + 4\pi r + 4\pi}{\pi r^2}\right) - 1\right) \cdot 100$$

$$= \left(\left(1 + \frac{4}{R} + \frac{4}{R^2}\right) - 1\right) \cdot 100$$

$$= \left(\frac{4}{R} + \frac{4}{R^2}\right) 100$$

$$= \boxed{\frac{400}{R} + \frac{400}{R^2}}$$

b) Let p = percent larger

$$p = \frac{400}{R} + \frac{400}{R^2}$$

c) domain (radius): 5–9 (inches)
range (area): 78.540 – 254.469 (inches²)

d) The graph represents areas of the pizzas in terms of the pizzas' radii. The data indicates the correct percentages when the areas are compared.

e) This problem is limited only by real-world circumstances because the size of the pizza able to be made depends upon the width and depth of the oven. Since there could not be an infinitely large oven, there cannot be an infinitely large pizza.

You’re Toast, Dude!

At the You’re Toast, Dude! Toaster Company, the weekly cost (in dollars) of producing x toasters is given by $C(x) = 4x + 1{,}400$.

1. Compute and interpret:

   a) $C(100)$

   b) $\frac{C(100)}{100}$

2. Use technology to produce a graph of the function $y = \frac{C(x)}{x},\ x > 0$, in an appropriate viewing window. Describe what the graph represents. Then describe all asymptotic behavior for this function and explain the meaning of this behavior in the given context.

3. Use the following two methods to find the number of toasters that must be produced in one week so that the average cost per toaster is $8:

   a) Use technology to locate the intersection of the graphs of two appropriately chosen equations.

   b) Set up an appropriate equation and solve algebraically.

4. How can $\frac{C(x)}{x}$ be written so that the horizontal asymptotic behavior described in number 2 above is more obvious? (Hint: Use the distributive property to divide each term of the numerator by x.) In what other ways does this new representation of the average cost function reveal insights into the behavior of this function for the given context?

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus:
(2A.10) Rational functions. The student formulates equations and inequalities based on rational functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

(A) use quotients of polynomials to describe the graphs of rational functions, predict the effects of parameter changes, describe limitations on the domains and ranges, and examine asymptotic behavior.

(B) analyze various representations of rational functions with respect to problem situations.

(C) determine the reasonable domain and range values of rational functions, as well as interpret and determine the reasonableness of solutions to rational equations and inequalities.

(D) determine the solutions of rational equations and inequalities using graphs, tables, and algebraic methods.

Scaffolding Questions:

- What does x represent?
- What does y, or $C(x)$, represent?
- What do you know if you divide the total cost by the number of toasters?
- As we move along the graph toward the y-axis, what is happening to x? To y?
- As we move along the graph away from the y-axis, what is happening to x? To y?
- Is there a number below which the values of y will never fall?
- Does $8 represent a value of x or y?
- How can we solve an equation with a rational expression?
- In rewriting the function, we can see that each average cost is $4 plus some quantity given by $\frac{1,400}{x}$. What is the role of $4 in the cost equation? How can we interpret the quantity given by $\frac{1,400}{x}$?

Sample Solutions:

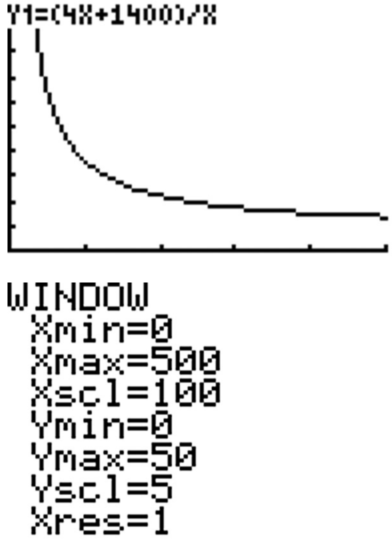
1. a) $C(100) = 4(100) + 1,400 = 1,800$

   The cost of producing 100 toasters in a week is $1,800.

   b) $\frac{C(100)}{100} = \frac{1,800}{100} = 18$

   When the toaster company produces 100 toasters in a week, the average cost per toaster is $18.

2.



The graph above shows the average cost per toaster, *Y1*, as a function of *X*, the number of toasters produced in one week. Although many choices for the window would be appropriate, the window used for this graph above is given below the graph. It was selected because for this problem situation, the x represents the number of toasters.

Vertical asymptote: $x = 0$

As the number of toasters, x, nears 0, the average cost per toaster gets larger. For example, when $x =$ 10 toasters, the average cost per toaster is \$144. This amount is more than the average cost per toaster when producing 100 toasters per week.

Horizontal asymptote: $y = 4$

As the number of toasters, x, becomes larger and larger, the average cost per toaster nears \$4. For example, when $x = 5{,}000$ toasters, the average cost per toaster is \$4.28.

(F) analyze a situation modeled by a rational function, formulate an equation or inequality composed of a linear or quadratic function, and solve the problem.

Additional Algebra II TEKS:
(2A.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

The student is expected to:

(A) identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations.

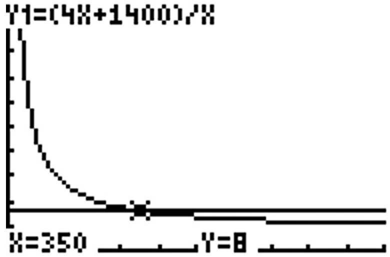
Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

3. a) 

The viewing window on the left shows that the intersection of $y = \dfrac{4x+1,400}{x}$ and $y = 8$ is the point (350, 8). So, when 350 toasters are produced a week, the average cost per toaster is $8.

b) $$\frac{4x+1,400}{x} = 8$$
$$4x + 1,400 = 8x$$
$$4x = 1,400$$
$$x = 350 \text{ toasters}$$

4. $$y = \frac{4x+1,400}{x} = \frac{4x}{x} + \frac{1,400}{x} = 4 + \frac{1,400}{x}$$

Notice that the new representation of this function, $4 + \dfrac{1,400}{x}$, shows that the average cost will always be $4 plus the quantity $\dfrac{1,400}{x}$. In other words, the average cost is $4 (the slope of the cost function, or the variable cost per toaster) plus an amount determined by dispersing the $1,400 (the *y*-intercept of the cost function, or the fixed costs) that, before the "first" toaster is made, must be expended equally among the total number of toasters produced, *x*. Since $\dfrac{1,400}{x}$ gets smaller and smaller as *x* gets larger and larger, $4 + \dfrac{1,400}{x}$ gets closer and closer to 4 as *x* gets larger and larger. This is horizontal asymptotic behavior with horizontal asymptote *y* = 4.

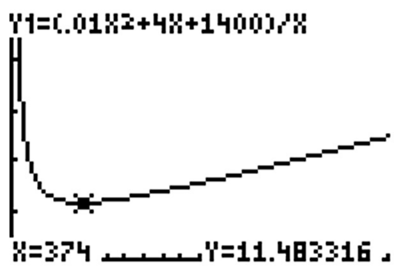
Extension Question:

- Use technology to produce a graph of the average cost function $y = \dfrac{C(x)}{x}$, $x > 0$, if $C(x) = 0.01x^2 + 4x + 1{,}400$. Use the viewing window described below:

```
WINDOW
 Xmin=0
 Xmax=2000
 Xscl=100
 Ymin=0
 Ymax=50
 Yscl=10
 Xres=1
```

Describe all asymptotic behavior for this function and explain the meaning of this behavior in the given context. Point out the similarities and differences in comparing the graph of this average cost function to the graph of the average cost function from the original problem.

The graph of this average cost function in the given viewing window is shown below:



The graphs are similar in that they both have a vertical asymptote of x = 0. In other words, as the number of items (in this case, toasters) being produced nears 0, the average cost per item gets larger.

The graphs differ in the following way: Whereas the original average cost function has a horizontal asymptote of y = 4 (meaning the average cost per unit nears 4 as the number of units becomes larger and larger), this function has no horizontal asymptote. As shown in the viewing window above, the average cost per unit initially decreases to a minimum of about \$11.48 when 374 units are produced.

For x > 374, as the number of units grows larger and larger, the average cost per unit grows larger and larger as well. Insights into the behavior of this function can be gained, once again, by algebraically re-expressing this function like we did in the original problem:

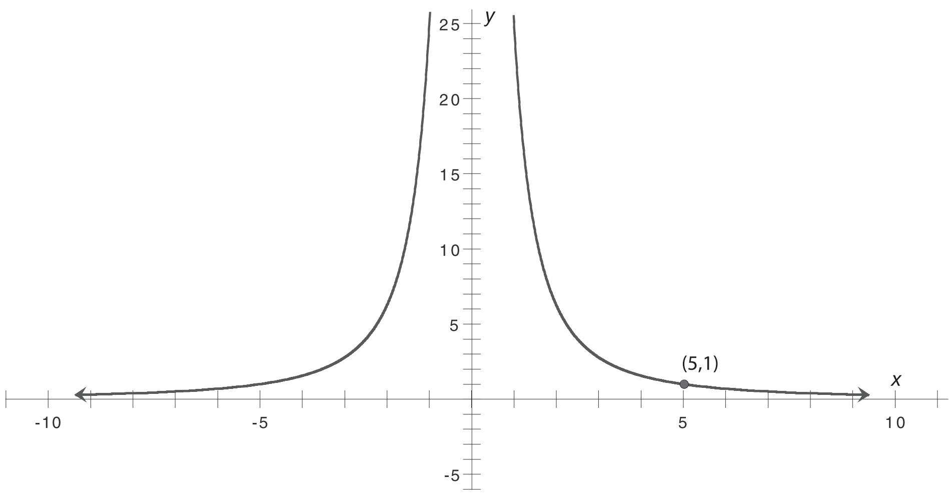
$$y = \frac{0.01x^2 + 4x + 1,400}{x} = \frac{0.01x^2}{x} + \frac{4x}{x} + \frac{1,400}{x} = 0.01x + 4 + \frac{1,400}{x}$$

Notice that the new representation of this average cost function shows that the average cost will eventually behave much like the linear function y = 0 .01x + 4 when x becomes very large (since the expression $\frac{1,400}{x}$ *becomes less and less significant as x grows larger and larger). In fact, the line y =0 .01x + 4 is a slant or oblique asymptote for the graph of this average cost function.*

What's My Equation?

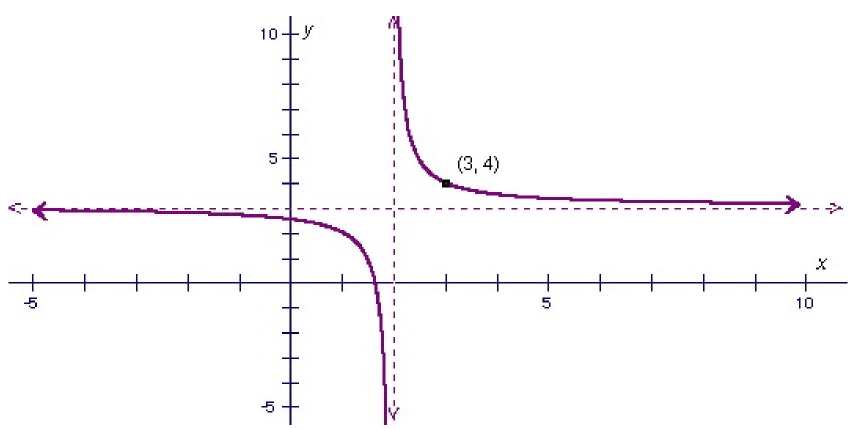
For each problem below, you are given the graph of a rational function along with some additional information about that function. Carefully analyze each graph and the additional information; then supply an appropriate function rule for each rational function whose graph is shown. Describe the domain and range for each function.

1.



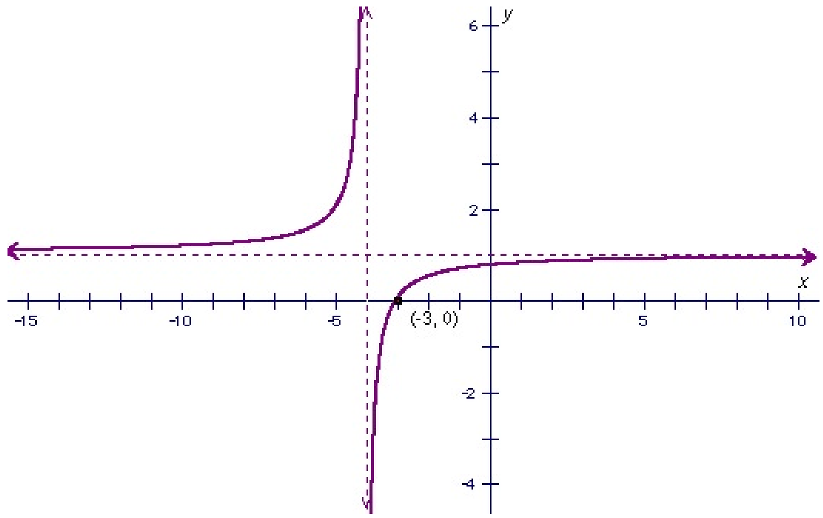
Additional information: *y* is inversely proportional to the square of *x*.

2.



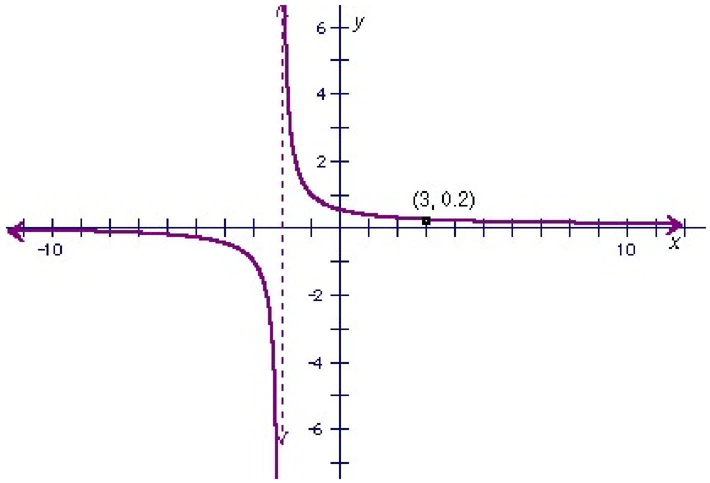
Additional information: The parent function for this graph is $y = \dfrac{1}{x}$.

3.



Additional information: The parent function for this graph is $y = \frac{1}{x}$.

4.



Additional information: If not for the “hole” in the graph at (3, 0.2), this graph could be generated with just a horizontal shift using $y = \frac{1}{x}$ as the parent function.

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus:
(2A.10) Rational functions. The student formulates equations and inequalities based on rational functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

(A) use quotients of polynomials to describe the graphs of rational functions, predict the effects of parameter changes, describe limitations on the domains and ranges, and examine asymptotic behavior.

(C) determine the reasonable domain and range values of rational functions, as well as interpret and determine the reasonableness of solutions to rational equations and inequalities.

Scaffolding Questions:

- What does the equation look like if *y* is proportional to *x*? What does the equation look like if *y* is inversely proportional to *x*?
- How can we use the given point to find the unknown parameters that affect the graph of the parent function?
- What does the graph of $y = \frac{1}{x}$ look like? What transformations of $y = \frac{1}{x}$ would result in the given graph?
- What algebraic manipulation of an expression that defines the function results in a vertical shift? A horizontal shift? A reflection across the *x*-axis? A reflection across the *y*-axis?
- How could we use the given point to help us find an equation?
- How could we use the given point to confirm that our equation is correct?
- What is the reflection of $y = \frac{1}{x}$ across the *x*-axis? What is the reflection of $y = \frac{1}{x}$ across the *y*-axis? Compare each reflection and discuss the reasons for their differences and similarities.
- What would the equation be if we "filled in the hole?" What does the "hole" have to do with the domain? With the equation?

Sample Solutions:

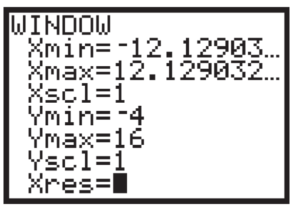
1. Since y is inversely proportional to the square of x, then $y = \frac{k}{x^2}$ for some non-zero constant, k.

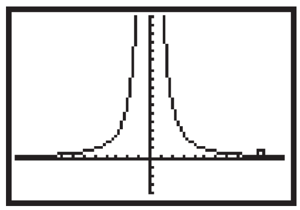
   Substituting the given point, we have $1 = \frac{k}{5^2}$ or $k = 25$.

   So, the rational function whose graph is shown is given by the equation

$$y = \frac{25}{x^2}$$

   This rule can be verified by using a graphing calculator.





   The domain is the set of all real numbers except 0.

   The range is the set of all real numbers greater than 0.

2. Starting with the parent function $y = \frac{k}{x}$, a horizontal shift to the right of 2 units and a vertical shift up of 3 units will result in the given graph. The resulting equation will be

$$y = \frac{k}{x-2} + 3$$

   The point (3, 4) lies on the graph. Substitute 3 for x and 4 for y and solve for k.

$$4 = \frac{k}{3-2} + 3$$

$$4 = k + 3$$

$$k = 1$$

$$y = \frac{1}{x-2} + 3 \text{ or } y = \frac{1 + 3(x-2)}{x-2} = \frac{3x-5}{x-2}$$

$$y = \frac{3x-5}{x-2}$$

Additional Algebra II TEKS:
(2A.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

The student is expected to:

(A) identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations.

(2A.4) Algebra and geometry. The student connects algebraic and geometric representations of functions.

The student is expected to:

(B) extend parent functions with parameters such as a in $f(x)$ = a/x and describe the effects of parameter changes on the graph of parent functions.

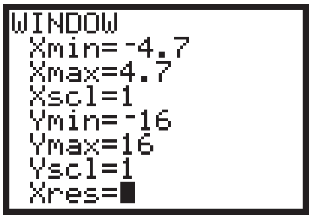
Connection to TAKS:

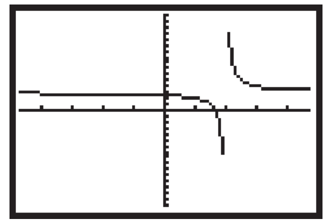
Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

The rule can be verified by using a graphing calculator.





The domain is the set of all real numbers except 2.

The range is the set of all real numbers except 3.

3. Starting with the parent function $y=\frac{k}{x}$, a reflection over the *x*-axis or a reflection of the *y*-axis followed by a horizontal shift to the left of 4 units and a vertical shift up of 1 unit will result in the given graph. The resulting function rule will be

$$y=\frac{k}{x+4}+1$$

To determine the value of *k* using the point (-3, 0), substitute -3 for *x* and 0 for *y*.

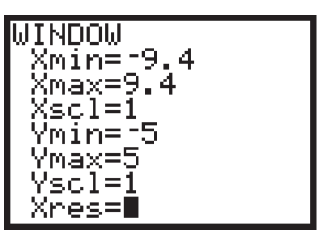
$$0=\frac{k}{-3+4}+1$$

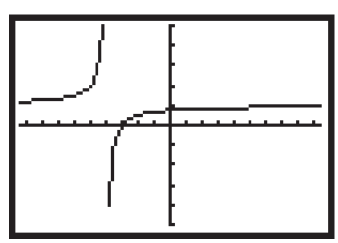
$$k=-1$$

$$y=\frac{-1}{x+4}+1 \text{ or } y=\frac{-1+1(x+4)}{x+4}=\frac{x+3}{x+4}$$

Substitution of the point (-3, 0) into this rule is further confirmation. Also, the point (1, 1) on the parent function will shift to (-5, 2) after the three transformations described above.

The graph on the calculator also matches the original graph.





The domain of the function is the set of all real numbers except -4. The range is the set of all real numbers except 1.

4. Ignoring the "hole" and starting with the parent function $y = \frac{1}{x}$, a horizontal shift to the left of 2 units would result in the given graph. The resulting equation would be

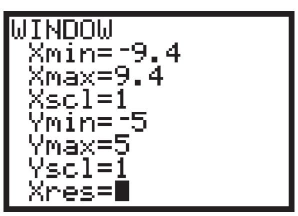
$$y = \frac{1}{x+2}$$

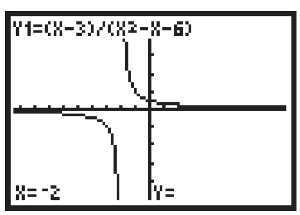
However, the graph of $y = \frac{1}{x+2}$ does not have a "hole" at (3, 0.2). In order to "remove" this point from the graph, we need to "remove" $x = 3$ from the domain of the function. This suggests a need to have the factor $x - 3$ in the denominator. In order for the rest of the graph to remain unchanged, we can introduce the factor $x - 3$ in the numerator as well. Another way to think about this change in the equation is to think about multiplying $\frac{1}{x+2}$ by the expression $\frac{x-3}{x-3}$, which is equivalent to 1 for all $x \neq 3$. The resulting equation will be

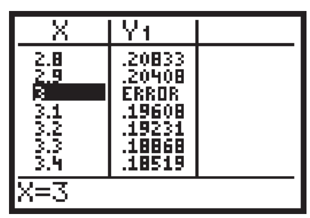
$$y = \frac{1}{x+2} \cdot \frac{x-3}{x-3} = \frac{x-3}{(x+2)(x-3)} \text{ or } \frac{x-3}{x^2-x-6}$$

The domain of this function is the set of all real numbers except -2 and 3. The range is the set of all real numbers except 0 and 0.2. 0.2 is the value of $y = \frac{1}{x+2}$ when $x = 3$.

The calculator graph and table verifies the accuracy of this rule, domain, and range.

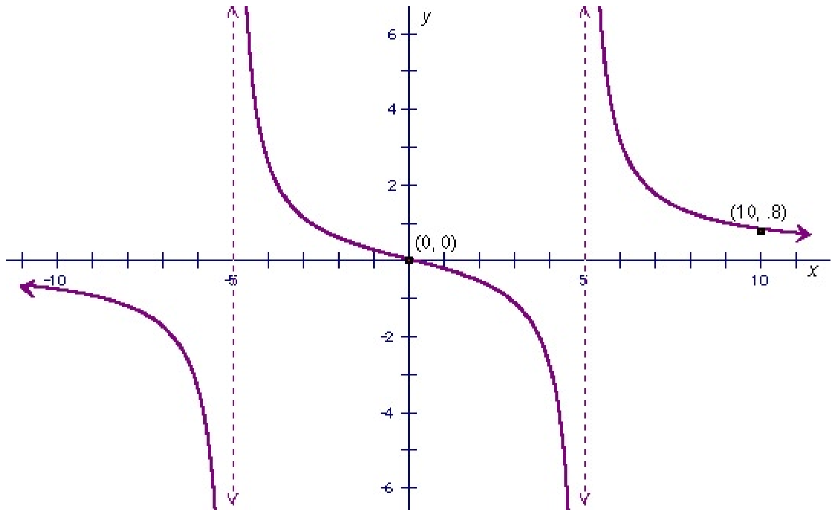






Extension Questions:

- Carefully analyze the following graph and the additional information given below the graph, then supply an appropriate rule for the rational function whose graph is shown. Describe the domain and range of the function.



Additional information: There are two vertical asymptotes; their equations are $x = 5$ and $x = -5$. There is a horizontal asymptote; its equation is $y = 0$.

With the two vertical asymptotes, x = 5 and x = -5, the denominator of this rational function must have the following factors: x – 5 and x + 5. With the horizontal asymptote y = 0, the degree of the numerator is less than the degree of the denominator. If we assume that the degree of the denominator is 2, then the degree of the numerator is either 0 or 1. The single x-intercept, (0, 0), implies that the degree is 1 and that the numerator is of the form kx, for some constant k. Therefore the rational function whose graph is shown has an equation of the form

$$y = \frac{kx}{(x-5)(x+5)} \text{ or } y = \frac{kx}{x^2 - 25}$$

All we need to do now is determine a value for k so that the graph of this function passes through the point (10, .8).

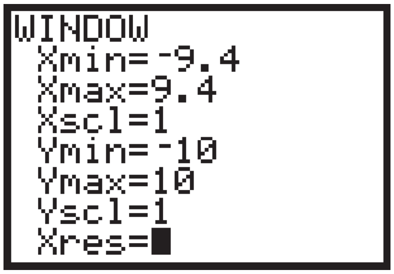
$$.8 = \frac{k(10)}{10^2 - 25}$$

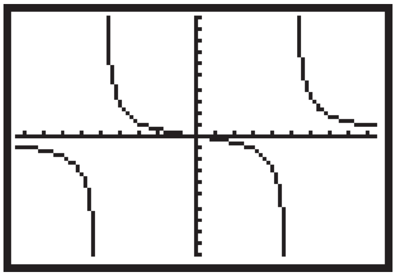
$$.8(75) = 10k$$

$$k = \frac{60}{10} = 6$$

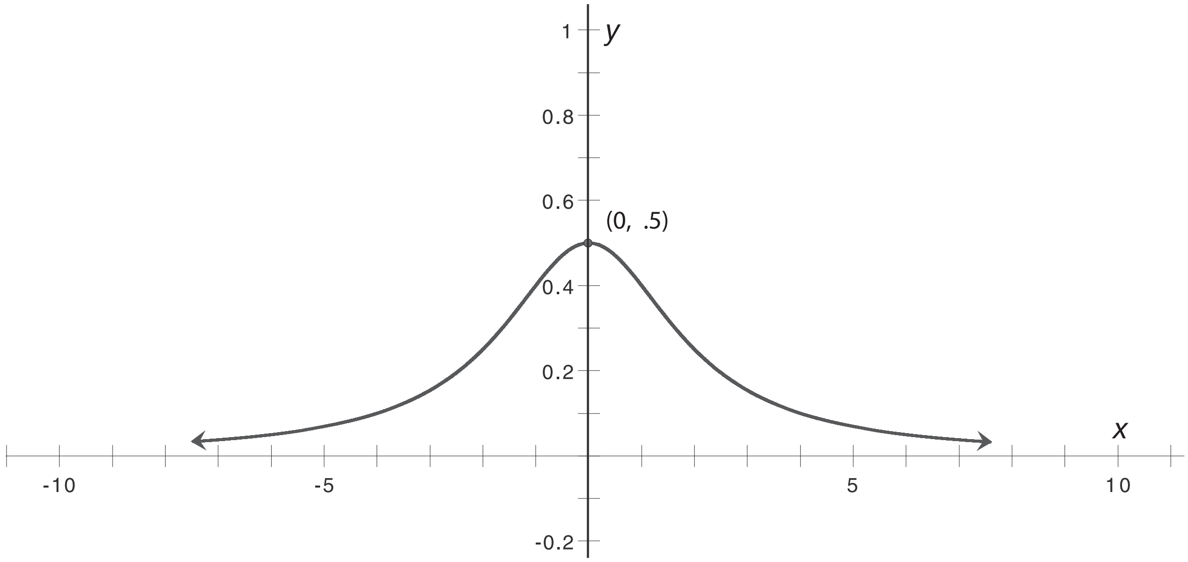
The rational function whose graph is shown is given by the equation $y = \dfrac{6x}{x^2 - 25}$

The domain is all real numbers except 5 and -5. The range is the set of all real numbers.





- Carefully analyze the following graph and the additional information given below the graph; then supply an appropriate equation for the rational function whose graph is shown. Describe the domain and range.



Additional information: The denominator is quadratic and has the following roots: 2*i* and -2*i*. There is a horizontal asymptote; its equation is *y = 0.*

Since 2i and -2i are roots of the denominator, x – 2i and x + 2i are factors of the denominator. Since the denominator is quadratic, we'll assume that the denominator is simply the product of these two factors:

$$(x - 2i)(x + 2i) = x^2 - 4i^2 = x^2 + 4$$

With the horizontal asymptote y = 0, the degree of the numerator is less than the degree of the denominator—either 0 or 1. With no x-intercept, the degree of the numerator can't be 1. So the numerator is a constant and the rational function has the following form:

$$y = \frac{k}{x^2 + 4} \text{, for some constant } k$$

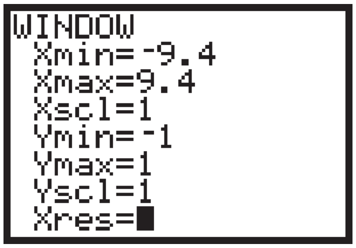
Using the y-intercept (0, .5), we can solve the following equation to find k:

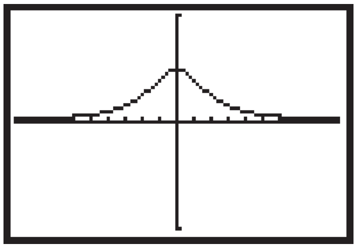
$$0.5 = \frac{k}{0^2 + 4}$$
$$k = .5(4) = 2$$

So the rational function whose graph is shown is given by the equation

$$y = \frac{2}{x^2 + 4}$$

The graph can be verified using a graphing calculator.





The domain is the set of all real numbers. The range is the set of all real numbers $0 \le y \le 0.5$ *.*

Chapter 8:
Conics

Contemplating Comets

As a comet moves through space, its path may be that of a conic section. All comets travel along paths that have their sun at one focus of the conic section. Barbara's science fair project focuses on the paths of two particular comets: Alphazoid and Betastar, in the Megacenturian solar system found in the Omega galaxy. However, she is having difficulty distinguishing the shapes of the paths of the comets. Barbara has asked you to help her match the correct conic section with each equation. The equations of the paths of the two comets are given below in billions of miles.

Alphazoid: $3x^2 + 2y^2 - 12x - 4y - 136 = 0$

Betastar: $16y^2 - 9x^2 + 36x - 32y - 164 = 0$

1. Complete the square to determine whether the path of the equation of the given comet is circular, parabolic, elliptical, or hyperbolic. Graph each comet's path on a separate grid. If the path is circular, give its center and radius. If it is parabolic, give its vertex and focus. If it is elliptical or hyperbolic, give its center and foci.
2. Since the comets travel along paths that have their sun as a focus, what are the coordinates of their sun?
3. Could these two comets collide? If so, where? Explain your reasoning.

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus:
(2A.5) Algebra and geometry. The student knows the relationship between the geometric and algebraic descriptions of conic sections.

The student is expected to:

(B) sketch graphs of conic sections to relate simple parameter changes in the equation to corresponding changes in the graph.

(C) identify symmetries from graphs of conic sections.

(D) identify the conic section from a given equation.

(E) use the method of completing the square.

Additional Algebra II TEKS:
(2A.3) Foundations for functions. The student formulates systems of equations and inequalities from problem situations, uses a variety of methods to solve them, and analyzes the solutions in terms of the situations.

The student is expected to:

(A) analyze situations and formulate systems of equations in two or more unknowns or inequalities in two unknowns to solve problems.

Scaffolding Questions:

- How can you determine which conic section the equation represents before it is transformed into standard form?
- How do you find the missing terms when you complete the square?
- How does the standard form of the equation help you find the center, focus, vertices, and major and minor axes lengths?
- How do you determine the coordinates of the foci given the information in standard form?

Sample Solutions:

1. In order to identify the conic section formed by the given equation, complete the square for the x- and y-terms and transform the equation into standard form. The following equation is calculated for Alphazoid.

$$3x^2 + 2y^2 - 12x - 4y - 136 = 0$$
$$3x^2 - 12x + 2y^2 - 4y = 136$$
$$3(x-2)^2 + 2(y-1)^2 = 150$$
$$\frac{3(x-2)^2}{150} + \frac{2(y-1)^2}{150} = 1$$
$$\frac{(x-2)^2}{50} + \frac{(y-1)^2}{75} = 1$$

The equation represents an ellipse with a major vertical axis. The center is (2, 1).

To find the distance from the center to the foci, use $c^2 = a^2 - b^2$, where a is the length of the semi-major axis, b is the length of the semi-minor axis, and c is the distance from the center to the foci.

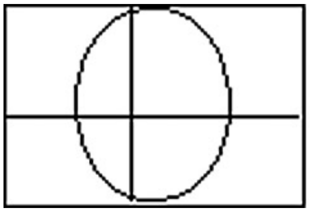
$c^2 = 75 - 50$ $\qquad a^2 = 75$ $\qquad b^2 = 50$
$c^2 = 25, c = 5$ $\qquad a \approx 8.7$ $\qquad b \approx 7.1$

If the center of the ellipse is the point (h, k), the coordinates of the vertices on the semi-major axis are ($h, k + a$) and ($h, k – a$) or (2, 9.7) and (2, -7.7).

The coordinates of the vertices on the semi-minor axis are ($h + b, k$) and ($h – b, k$) or (9.1, 1) and (-5.1, 1).

The coordinates of the foci are ($h, k + c$) and ($h, k – c$) or (2, 6) and (2, -4).

The following graph shows the path of the comet Alphazoid.



In order to identify the conic section formed by the second given equation, complete the square for the x- and y-terms and transform the equation into standard form. The following equation is calculated for Betastar.

$$16y^2 - 9x^2 + 36x - 32y - 164 = 0$$
$$16y^2 - 32y - 9x^2 + 36x = 164$$
$$16(y^2 - 2y + 1) - 9(x^2 - 4x + 4) = 164 + 16 - 36$$
$$16(y - 1)^2 - 9(x - 2)^2 = 144$$
$$\frac{16(y - 1)^2}{144} - \frac{9(x - 2)^2}{144} = 1$$
$$\frac{(y - 1)^2}{9} - \frac{(x - 2)^2}{16} = 1$$

The equation represents a hyperbola with the transverse axis $x = 2$. The center (h, k) is (2, 1).

To find the distance from the center to the foci, use $c^2 = a^2 + b^2$.

$a^2 = 9$	$b^2 = 16$	$c^2 = 9 + 16$
$a = 3$	$b = 4$	$c^2 = 25$, $c = 5$

(B) use algebraic methods, graphs, tables, or matrices, to solve systems of equations or inequalities.

Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

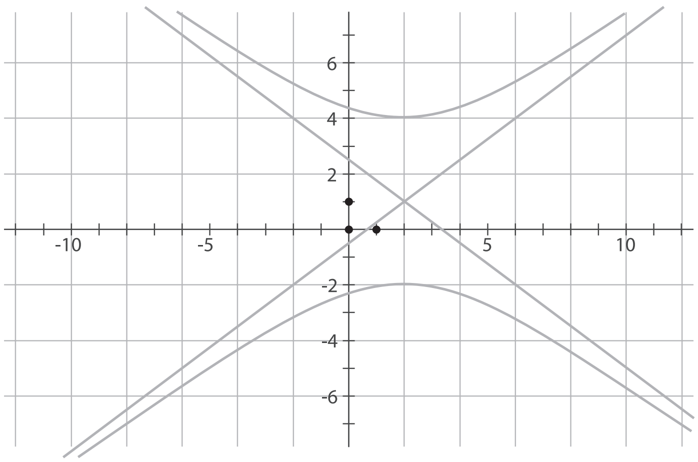
Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

The coordinates of the foci are (*h, k + c)* and (*h, k – c*) or (2, 6) and (2, -4).

The coordinates of the vertices on the transverse axis are (*h, k + a)* and (*h, k – a)* or (2, 4) and (2, -2).

The asymptotes are $y = \frac{3}{4}(x - 2) + 1$ and $y = -\frac{3}{4}(x - 2) + 1$.

The following graph shows the possible paths of the comet Betastar.



2. Both conic sections have foci at (2, 6) and (2, -4). The position of their common sun is (2, 6) if Betastar travels the northern branch of the hyperbolic curve. If Betastar travels on the southern branch of the hyperbolic curve, the coordinates of their sun are (2, -4).

3. To determine if the two comets collide, solve the system of equations for the comets.

Alphazoid: $3x^2 + 2y^2 - 12x - 4y - 136 = 0$
Betastar: $16y^2 - 9x^2 + 36x - 32y - 164 = 0$

Reorder the terms in the equations.

Alphazoid: $3x^2 - 12x + 2y^2 - 4y - 136 = 0$
Betastar: $-9x^2 + 36x + 16y^2 - 32y - 164 = 0$

Multiply the equation for Alphazoid by 3.

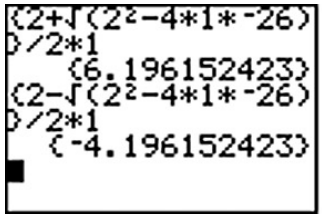
Alphazoid: $9x^2 - 36x + 6y^2 - 12y - 408 = 0$
Betastar: $-9x^2 + 36x + 16y^2 - 32y - 164 = 0$

Add the two equations together. The result is $22y^2 - 44y - 572 = 0$.

Divide by 22.

$$y^2 - 2y - 26 = 0$$

Use the quadratic formula to solve, where $a = 1$, $b = -2$, and $c = -26$.



The values for y (rounded to the nearest tenth) are $y = 6.2$ or $y = -4.2$.

To find the value of x, substitute each value of y into one of the original equations. First, $y = 6.2$ and the equation for Alphazoid is used.

$$3x^2 - 12x + 2y^2 - 4y - 136 = 0$$
$$3x^2 - 12x + 2(6.2)^2 - 4(6.2) - 136 = 0$$
$$3x^2 - 12x + 76.88 - 24.8 - 136 = 0$$
$$3x^2 - 12x - 83.92 = 0$$

Use the quadratic formula to solve for x, where $a = 3$, $b = -12$, and $c = -83.92$.

$$x = \frac{12 \pm \sqrt{(-12)^2 - 4(3)(-83.92)}}{2(3)}$$
$$x = \frac{12 \pm \sqrt{1151.04}}{2(3)}$$
$$x = 7.7 \quad \text{or} \quad x = -3.7$$

(rounded to the nearest tenth)

If Betastar travels the northern branch of the hyperbolic path, the possible collision points are (7.7, 6.2) or (-3.7, 6.2).

Next, using $y = -4.2$, solve for x. To find the value of x, substitute each value of y into one of the original equations. The equation for Alphazoid is again used.

$$3x^2 - 12x + 2y^2 - 4y - 136 = 0$$
$$3x^2 - 12x + 2(-4.2)^2 - 4(-4.2) - 136 = 0$$
$$3x^2 - 12x + 35.28 - 16.8 - 136 = 0$$
$$3x^2 - 12x - 83.92 = 0$$

Use the quadratic formula to solve for *x*, where *a* = 3, *b* = -12, and *c* = 83.92.

$$x = \frac{12 \pm \sqrt{(-12)^2 - 4(3)(-83.92)}}{2(3)}$$

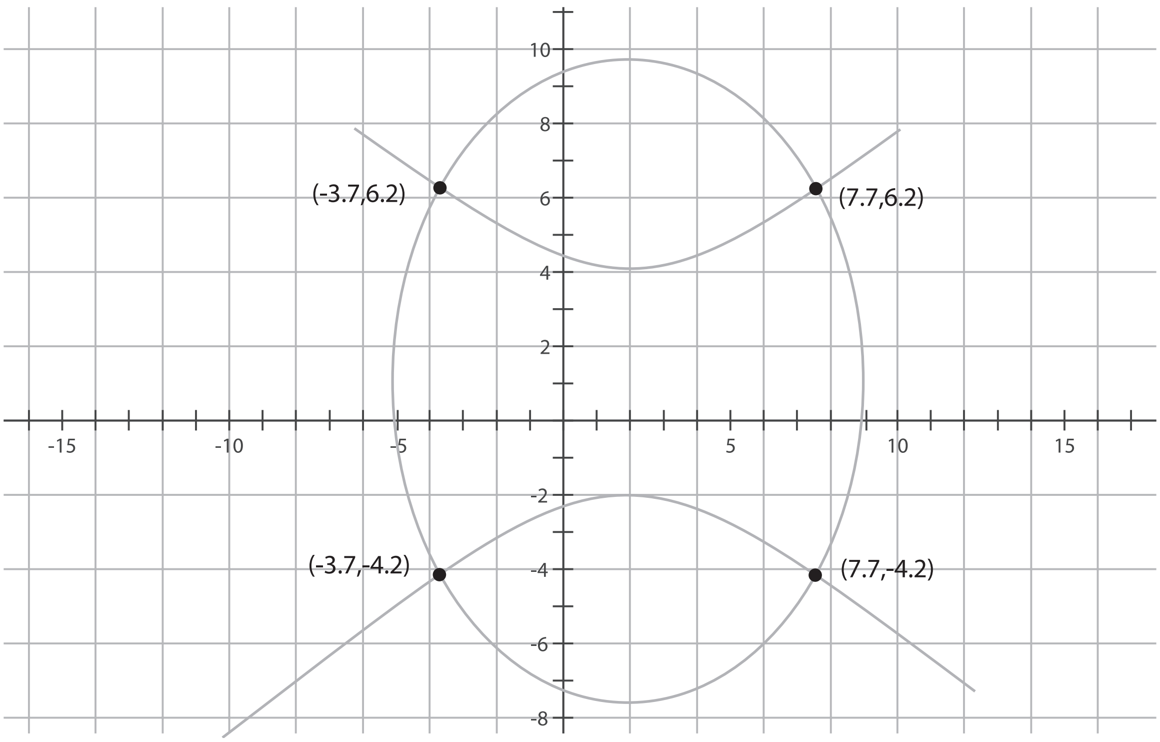
$$x = \frac{12 \pm \sqrt{1151.04}}{2(3)}$$

$$x = 7.7 \quad \text{or} \quad x = -3.7$$

(rounded to the nearest tenth)

If Betastar travels the northern branch of the hyperbolic path, the possible collision points are (7.7, -4.2) or (-3.7, -4.2).

The paths of the two comets cross each other, so there *is* a possibility of a collision.



Extension Questions:

- Barbara's research on the comet Alphazoid led to an interesting discovery. Several hundred years ago, the comet's path was represented by the equation $x^2 + 2x + y^2 + 10y = 23$. Describe the shape and key elements of the comet's path.

  The comet's path is circular because the coefficients on the x and y squared terms are equal. Complete the square to determine the radius and center.

  $$x^2 + 2x + 1 + y^2 + 10y + 25 = 23 + 1 + 25$$

  .

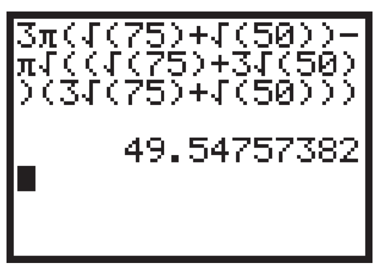
  $$(x + 1)^2 + (y + 5)^2 = 49$$

  The center is (-1, -5) and the radius is 7.

- The formula $C = 3\pi(a + b) - \pi\sqrt{(a + 3b)(3a + b)}$ can be used to approximate the circumference *C* of an ellipse with major axis *a* and minor axis *b*. Determine the circumference of the Alphazoid orbit.

  The values for a and b determined in problem 1 and 2 are $a = \sqrt{75}$ and $b = \sqrt{50}$ *. Substitute these values into the formula.*

  $$C = 3\pi(a + b) - \pi\sqrt{(a + 3b)(3a + b)}$$
  $$C = 3\pi(\sqrt{75} + \sqrt{50}) - \pi\sqrt{(\sqrt{75} + 3\sqrt{50})(3\sqrt{75} + \sqrt{50})}$$



  The area of the ellipse is approximately 49.55 million miles.

Lost in Space

The conic sections play a fundamental role in space science. Johannes Kepler was the first European to think that planets moved in elliptical rather than circular orbits around the sun. His detailed study of planetary motion led to the Laws of Planetary Motion, which are known today as Kepler's Laws. The laws apply to any celestial object that orbits another object under the influence of gravity. Two of Kepler's Laws are shown below.

- Kepler's First Law: The orbit of each planet is an elliptical path with the Sun at one focus of the ellipse.
- Kepler's Third Law: For a planet, the cube of the orbit's semi-major axis (measured in astronomical units–AU) is equal to the square of the planet's orbital period (measured in Earth years).

1. The point at which a planet is at its closest to the sun is called *perihelion*. At perihelion, a newly discovered planet, Alpha-7, is 28.6 million miles from the sun. *Aphelion* is the point that a planet is farthest from the sun. Alpha-7 is 43.4 million miles from the sun at aphelion.
   a. Use the given information to sketch and label a model of Alpha-7's orbit. Assume the foci lie on the *x*-axis, and the origin is the center of the ellipse.
   b. Determine the lengths of the major and semi-major axes of the elliptical orbit.
   c. Determine the sun's distance from the center of the ellipse.
   d. Explain how to determine the length of the semi-minor axis of Alpha-7's orbit. (Round to the nearest tenth.)
   e. Use Kepler's First Law to help determine the equation of Alpha-7's orbit. (Round to the nearest tenth.) Include the coordinates of the vertices of the ellipse, and of the sun. Justify your answers.
2. Use Kepler's Third Law to determine the length of Alpha-7's orbital cycle in Earth years. Round your answer to the nearest hundredth. Note: In order to use Kepler's Third Law, it will be necessary to express the length of the semi-major axis relative to that of the earth as follows:

   Earth axis = 93 million miles = 1 astronomical unit (AU)

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus:
(2A.5) Algebra and geometry. The student knows the relationship between the geometric and algebraic descriptions of conic sections.

The student is expected to:

(B) sketch graphs of conic sections to relate simple parameter changes in the equation to corresponding changes in the graph.

(C) identify symmetries from graphs of conic sections.

(D) identify the conic section from a given equation.

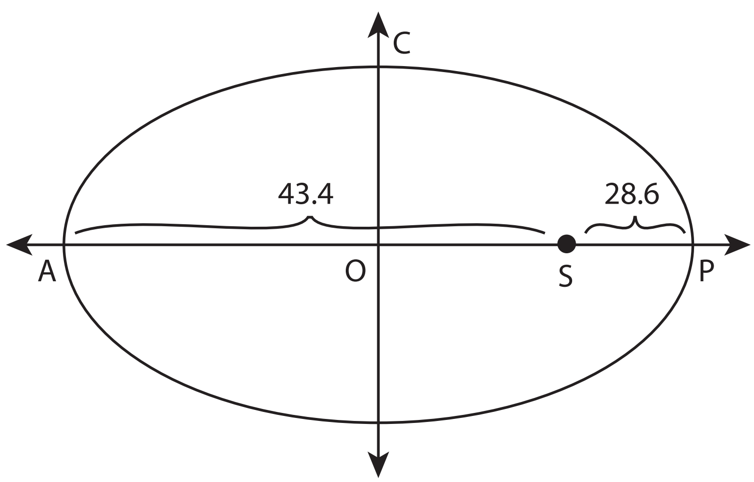
Additional Algebra II TEKS:
None

Scaffolding Questions:

- How many values are needed to write the equation of an ellipse whose center is at the origin?
- Where would you locate the sun in your sketch of the graph of the ellipse?
- How could you use the perihelion and aphelion distances to help determine the value of the semi-major axis?
- How can you use the perihelion and aphelion distances to help you find the distance from the origin to the sun?
- What is the relationship among the semi-major axis, center, semi-major axis, and the foci?

Sample Solutions:

1. a. Using the given information in millions of miles, the following is a sketch of Alpha-7's orbit. Assume the foci lie on the *x*-axis, and the origin is the center of the ellipse.



b. To determine the length of the major axis, add lengths AS and SP: 43.4 + 28.6 = 72 million miles. The length of the semi-major axis is half of AP, $\frac{1}{2}(72) = 36$ million miles, which equals the lengths of AO and OP.

c. The distance from the center of the ellipse is found using segment addition OS + SP = OP. It is known that OS + 28.6 = 36, therefore OS = 7.4 million miles.

d. The length of the semi-minor axis (OC) can be found using the following relationship. The standard form of an ellipse with center at the origin and foci on the *x*-axis is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where 2*a* is the length of the major axis and 2*b* is the length of the minor axis.

If the distance from the center to the foci is *c*, then the relationship between *a*, *b*, and *c* is $b^2 = a^2 - c^2$.

In this problem *a* = PO = 36

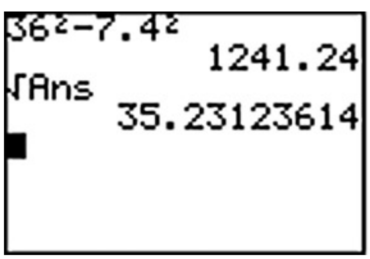
b = OC = unknown

c = OS = 7.4

Therefore,

$$PO^2 = OC^2 + OS^2$$

$$36^2 = OC^2 + 7.4^2$$



35.2 million miles = OC (rounded to the nearest tenth)

e. Kepler's First Law states the path of the orbit is an ellipse. Therefore the equation of Alpha-7's orbit is

$$\frac{x^2}{(36)^2} + \frac{y^2}{(35.2)^2} = 1$$

The coordinates of the vertices of the ellipse are at (-36, 0), (0, -35.2), (36, 0), (0, 35.2).

Connection to TAKS:

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

The coordinates of the sun (foci) are already known (from part c) because the value of c is 7.4: (-7.4, 0) and (7.4, 0).

The sun appears at (7.4, 0).

2. Kepler's Third Law states that the cube of a planet's semi-major axis is equal to the square of the planet's orbital period, where the semi-major axis and the orbital period are measured relative to those of Earth. Orbital period is the time to complete one orbit of the sun. Earth's orbital period is one year.

   In order to determine one of Alpha-7's orbital cycles in Earth years and use Kepler's Law, it will be necessary to express the length of the semi-major axis relative to that of the earth as follows: Earth's semi-major axis = 93 million miles = 1 astronomical unit (AU).

$$\frac{93 \text{ mil mi}}{1 \text{ AU}} = \frac{36 \text{ mil mi}}{x \text{ AU}}$$

$$93x = 36$$

$$x = \frac{36}{93} \text{ AU}$$

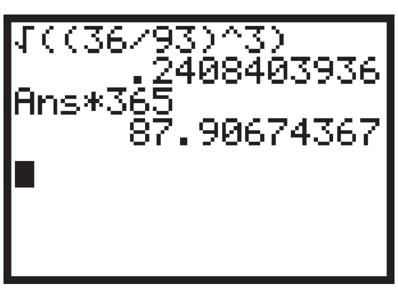
Kepler's Third Law states:

(Alpha-7's semi-major axis in AUs) ^{3} = (Alpha-7's orbital period) ^{2}
$(0.39)^3 = (x)^2$

$$\left(\frac{36}{93}\right)^3 = x^2$$

$$x = \sqrt{\left(\frac{36}{93}\right)^3}$$

This answer represents a portion of a year. To determine the number of days, the answer may be multiplied by 365, the number of days in a year.



The orbit period is about 0.241 years, or approximately 88 days.

Extension Questions:

- What is the equation of Alpha-7's orbit if the axes are positioned so that location of the sun is used as the origin instead of placing the center at the origin?

  The equation found in e was $\frac{x^2}{(36)^2}+\frac{y^2}{(35.2)^2}=1$ *. If the center of the origin is moved to the focus, (7.4, 0), the new equation becomes*

$$\frac{(x-7.4)^2}{(36)^2}+\frac{y^2}{(35.2)^2}=1$$

- The eccentricity, *e*, of an ellipse is a measurement of its "flatness." The formula *e* = *c/a* is used to find the eccentricity of an ellipse, where *c* is the distance from the center to a focus and *a* is the distance from the center to a vertex along the major axis.

  Consider the orbit of the asteroid Zeta around the sun. At perihelion, Zeta is approximately 2.75 billion miles from the sun. Aphelion is the point that a celestial body is at its greatest distance from the sun. Zeta is approximately 4.55 billion miles from the sun at aphelion. Round the answer to the nearest hundredth.

  a. Determine the eccentricity of the orbit. What does this value tell you about the characteristics of the elliptical orbit?
  b. Determine the equation of the Zeta's orbit.

  a. The length of the major axis of the ellipse is found by adding the closest distance and the farthest distance of the orbit. Given that the closest distance (perihelion) is 2.75 billion miles, and the farthest distance (aphelion) is 4.55 billion miles, the length of the major axis is 2.75 + 4.55, or 7.3 billion miles.

  The semi-major axis is half of that value, or 3.65 billion miles:

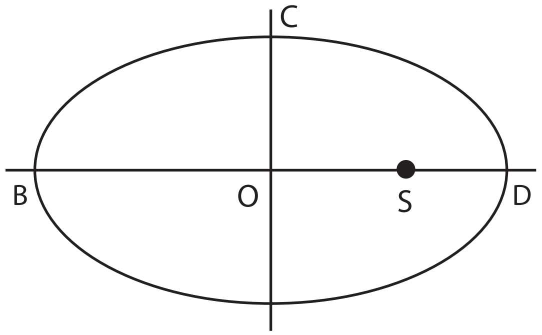
  a + c = 4.55 billion miles, therefore c = 0.9 billion miles

  The eccentricity of Zeta's orbit is determined by $e=\frac{c}{a}=\frac{0.9}{3.65}$.

  Rounded to the nearest hundredth, the eccentricity is approximately 0.25.

  The eccentricity of an ellipse can be any value between 0 and 1 because it is a ratio of two positive values, c and a and c < a. An ellipse with an *eccentricity close to 1 is very narrow because if* $\frac{c}{a}$ *is close to 1, c is close to a, and* $b=\sqrt{a^2-c^2}$ *must be very small. An ellipse with eccentricity close to 0 is almost a circle because if* $\frac{c}{a}$ *is almost equal to zero, a is much larger than c and* $b=\sqrt{a^2-c^2}$ *must be almost equal to a. Zeta's eccentricity is 0.25, which is closer to zero than to one; therefore it is fairly narrow.*

b.



BS = 4.55   SD = 2.75   BD = 7.3   OD = 3.65

$OC^2 = OD^2 - OS^2$
$OC^2 = 3.65^2 - 0.9^2$
$OC = 3.54$

The equation of Zeta's orbit is approximately $\frac{x^2}{(3.65)^2}+\frac{y^2}{(3.54)^2}=1$

Kalotonic Kaper

Lester Large is planning another diabolical plot to fell the superhero Heroic Horace. With the help of the brilliant scientist Dr. Madd, Lester has discovered a way to transform Kalotonic's destructive characteristics into a beam of light. Planning to stage a catastrophe involving Ms. Lana Lorrell, Lester will use a large hyperbolic mirrored surface to beam the Kalotonic onto Heroic Horace at the moment he rescues lovely Lana, thus disabling his archenemy.

Dr. Madd and Lester will project the beam from their hiding place in the vicinity of the building where Lana works. However, before Dr. Madd can proceed with the plan to lure Lana into the danger zone, he must determine the point at which the beam intersects the mirror.

The hyperbolic mirror is shaped like one branch of the hyperbola. It reflects any light directed toward one focus of the hyperbola through the other focus. Dr. Madd has devised his plan on a coordinate grid. He has placed the center of the hyperbola at the origin, and the vertex of the mirror's branch at (4, 0). The focus of the mirror's branch is at (5, 0), which is the danger zone for Lana and Heroic Horace. The equations of the asymptotes are $y = +\frac{3}{4}x$ and $y = -\frac{3}{4}x$. Dr. Madd and Lester's hiding place is at the coordinates (19, 6).

1. Using Dr. Madd's description, sketch a graph of the hyperbolic mirrored surface, its imaginary branch (i.e., the other branch of the mathematical hyperbola), the location of the hiding place, and the danger zone. Determine the location of the foci of both branches of the hyperbola.
2. Write a rule that models the mirrored surface.
3. Suppose Dr. Madd and Lester Large plan to direct the beam of Kalotonic from their location at (19, 6) to the focus of the imaginary branch. Where will the beam of Kalotonic intersect the hyperbolic mirror?

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus:
(2A.5) Algebra and geometry. The student knows the relationship between the geometric and algebraic descriptions of conic sections.

The student is expected to:

(B) sketch graphs of conic sections to relate simple parameter changes in the equation to corresponding changes in the graph.

(C) identify symmetries from graphs of conic sections.

Additional Algebra II TEKS:

(2A.8) Quadratic and square root functions. The student formulates equations and inequalities based on quadratic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

(B) analyze and interpret the solutions of quadratic equations using discriminants and solve quadratic equations using the quadratic formula.

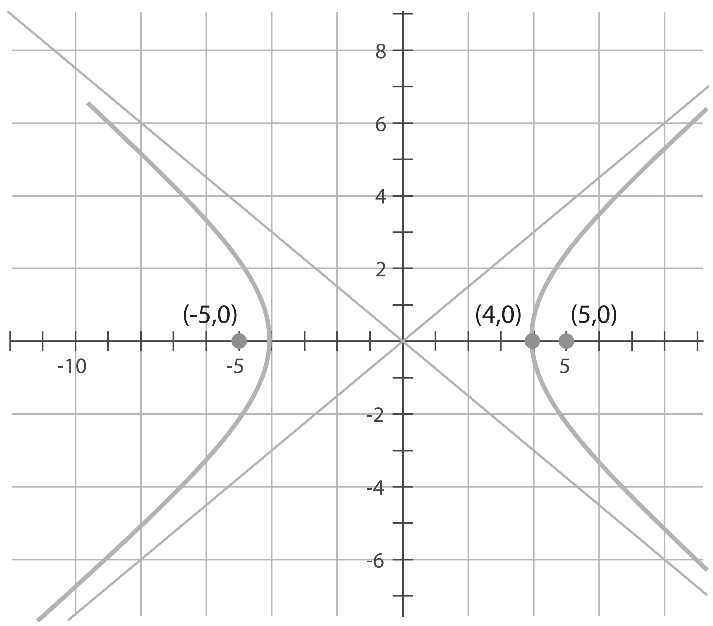
Scaffolding Questions:

- What is the center of the hyperbola?
- What is the standard form of an equation of a hyperbola with center (0, 0) and vertices on the *x*-axis?
- How can the equations of the asymptotes help you determine the values of *a* and *b* in your hyperbola equation?
- How is the location of the helicopter and the focus of the imaginary branch related to the beam of Kalotonic?

Sample Solutions:

1. The center of the hyperbola is at the origin (0, 0), the vertex of the mirror's branch at (4, 0), and the focus at (5, 0). Asymptotes are $y = +\frac{3}{4}x$ and $y = -\frac{3}{4}x$.

   The focus of the imaginary branch is at (-5, 0).



2. The standard form of an equation of a hyperbola with center (0, 0) and vertices on the *x*-axis is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The equations of the asymptotes are

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x$$

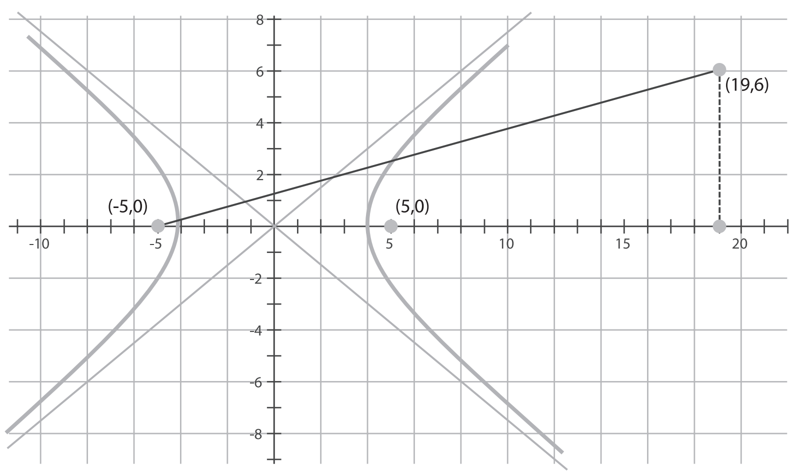
Therefore, $b = 3$ and $a = 4$.

The equation of the hyperbola is

$$\frac{x^2}{4^2} - \frac{y^2}{3^2} = 1$$

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

3. The Kalotonic beam will originate from the hiding place at point (19, 6) and go to the focus of the imaginary branch at (-5, 0).



The slope of the line joining these points is

$$\frac{6-0}{19-(-5)} = \frac{6}{24}$$

Connection to TAKS:

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

The equation of this beam is a line as follows:

$$y - 0 = \frac{6}{24}\ (x - (-5))$$
$$y = 0.25(x + 5)$$
$$y = 0.25x + 1.25$$

To find the point where the beam of Kalotonic will intersect the mirror, the system of equations (hyperbola and the beam) must be solved. The equation of the beam is solved for *y;* therefore, *x* can be found by substitution into the hyperbola equation.

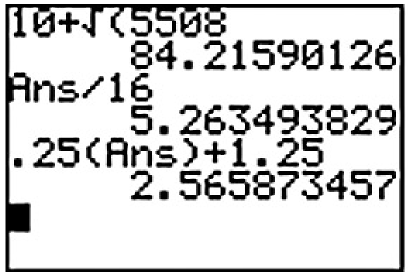
$$\frac{x^2}{4^2} - \frac{y^2}{3^2} = 1$$
$$\frac{x^2}{16} - \frac{(0.25x + 1.25)^2}{9} = 1$$
$$9x^2 - 16(0.25x + 1.25)^2 = 144$$
$$9x^2 - 16(0.0625x^2 + 0.625x + 1.5625) = 144$$
$$9x^2 - 1x^2 - 10x - 25 = 144$$
$$8x^2 - 10x - 169 = 0$$

Use the quadratic equation to solve,with *a* = 8, *b* = 10, *c* = -169.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(8)(-169)}}{2(8)}$$
$$x = \frac{10 \pm \sqrt{5508}}{16}$$
$$x \approx 5.3 \ \text{ or } \ x \approx -4.0$$

The positive solution, *x* = 5.3, is the only reasonable solution for this situation because only half of the hyperbola represents the mirrored surface.

This value is then substituted into the equation for the Kalotonic beam ($y = 0.25x + 1.25$). The solutions for x and then y are shown on the graphing calculator screen below.



The value of y is approximately 2.6.

The beam of Kalotonic will intersect the hyperbolic mirrored surface at about (5.3, 2.6).

Extension Question:

- Suppose Dr. Madd has erred in his calculations and placed the mirror at the wrong position. He determines he must relocate it 1 unit to the west of its current location. The location of his hiding place and his original target (the focus of the original imaginary branch of the hyperbola) will remain the same. Determine the coordinates of the point where the beam will intersect the hyperbolic mirror.

  If the mirror is relocated 1 unit west of its original position, the hyperbola is shifted 1 unit left along the x-axis. The center of the new hyperbolic mirror is (-1, 0). The vertex is located at (3, 0), and the focus at (4, 0).

  The equation of the hyperbolic mirror becomes $\frac{(x+1)^2}{16} - \frac{y^2}{9} = 1$

  The equation of the Kalotonic beam remains unchanged, y = 0.25x + 1.25.

  To locate the new point of intersection, first multiply the equation by 144, and then substitute y from the beam into the hyperbolic equation.

  $9(x+1)^2 - 16\,(.25x + 1.25)^2 = 144$
  $9x^2 + 18x + 9 - (x^2 + 10x + 25) = 144$
  $9x^2 + 18x + 9 - x^2 - 10x - 25 = 144$
  $8x^2 + 8x - 160 = 0$
  $x^2 + x - 20 = 0$
  $(x + 5)(x - 4) = 0$
  $x = -5, 4$

  Only x = 4 makes sense for this situation.

  To determine the value of y substitute 4 for x in the rule for the beam.

  $y = .25(4) + 1.25 = 2.25$

  The new point of intersection is (4, 2.25).