Chapter 3:
Linear Systems

A Linear Programming Problem: Parking at the Mall

A new mall with 2 major department stores and 55 specialty shops is being built. You are a subcontractor in charge of planning and building the parking lots for the mall. The planners provide you with the following information:

- The total number of parking spaces must range from 2,000 to 2,400 spaces.
- For every employee parking space there must be at least 9 public parking spaces.
- There must be at least 20 employee parking spaces per department store and 2 employee parking spaces per specialty shop.

You anticipate that building costs will be $580 per public parking space and $600 per employee parking place.

The mall planners expect that, during an average week, revenue (average customer spending) from each public parking space will be at least $1,000 and from each employee parking space will be at least $100.

Design a proposal to present to the mall planners showing the feasible numbers of public and employee parking spaces. How many parking spaces of each type should be built to minimize the cost of building the parking lot? How many parking spaces of each type should be built to maximize weekly revenue?

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus: (2A.3) Foundations for functions. The student formulates systems of equations and inequalities from problem situations, uses a variety of methods to solve them, and analyzes the solutions in terms of the situations.

The student is expected to:

(A) analyze situations and formulate systems of equations in two or more unknowns or inequalities in two unknowns to solve problems.

(B) use algebraic methods, graphs, tables, or matrices, to solve systems of equations or inequalities.

(C) interpret and determine the reasonableness of solutions to systems of equations or inequalities for given contexts.

Scaffolding Questions:

- What are the independent variables in this situation?
- Describe the restrictions (constraints) on the independent variables.
- How will you write these restrictions algebraically?
- What do these restrictions have to do with “the feasible region?”
- How will you go about graphing these restrictions?
- What does the cost of building the parking lot depend on? What function can you write for cost?
- What does the weekly revenue (average weekly customer spending per space) depend on? What function can you write for revenue?
- What is the Corner Principle for Linear Programming?
- What representations can you use to organize your proposal and answer the questions?

Sample Solutions:

Let x = the number of public parking spaces and y = the number of employee parking spaces.

To determine the feasible number of parking spaces to build, we need to describe the constraints on x and y.

Since the total number of parking spaces must be between 2,000 and 2,400,

$$2{,}000 \le x + y \le 2{,}400.$$

For every employee space there must be at least 9 public spaces, so

$$x \ge 9y \text{ or } y \le \frac{1}{9}x.$$

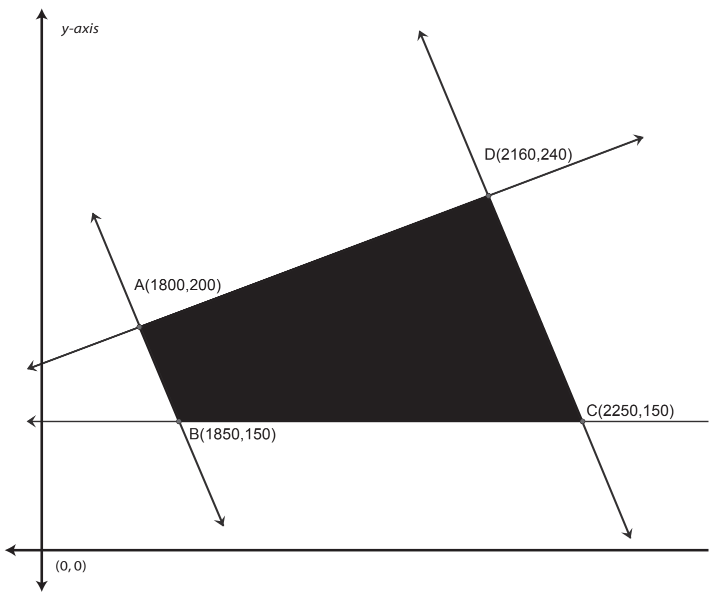
Finally, since we need at least 20 employee spaces for each of the 2 department stores and at least 2 employee spaces for each of the 55 specialty shops, we know that

$$y \geq 20(2) + 2(55)$$
$$y \geq 150.$$

The following restrictions are placed on the 2 variables:

$$y \geq 2{,}000 - x$$
$$y \leq 2{,}400 - x$$
$$y \leq \frac{1}{9}x$$
$$y \geq 150$$

The graph of the feasible region is shown below:



(Note: A good window for a calculator graph of this feasible region is $1{,}750 \leq x \leq 2{,}300, \quad 125 \leq y \leq 275$)

The points of intersection of the boundary lines are found by solving the systems.

Point A:

$$\begin{cases} x + y = 2{,}000 \\ x = 9y \end{cases} \Rightarrow \quad 9y + y = 2{,}000$$

$$10y = 2{,}000$$
$$y = 200$$
$$x = 9 \bullet 200$$
$$= 1{,}800$$

Additional Algebra II TEKS:
(2A.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

The student is expected to:

(A) identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations.

Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 4: The student will formulate and use linear equations and inequalities.

Point B:

$$\begin{cases} x + y = 2{,}000 \\ y = 150 \end{cases} \Rightarrow x + 150 = 2{,}000$$

$$x = 1{,}850$$
$$y = 150$$

Point C:

$$\begin{cases} x + y = 2{,}400 \\ y = 150 \end{cases} \Rightarrow x + 150 = 2{,}400$$

$$x = 2{,}250$$
$$y = 150$$

Point D:

$$\begin{cases} x + y = 2{,}400 \\ x = 9y \end{cases} \Rightarrow 9y + y = 2{,}400$$

$$10y = 2{,}400$$
$$y = 240$$
$$x = 9 \bullet 240$$
$$= 2{,}160$$

All points in the feasible region and on the boundary of the region with integer coordinates would be feasible numbers of public and employee parking spaces.

To predict the minimum cost of building the parking lot, we need to write a cost function, *C*. Since it costs \$580 per public space and \$600 per employee space, the cost function is given by

$$C = C(x, y) = 580x + 600y.$$

The Corner Principle in Linear Programming tells us that the extreme (minimum and maximum) will occur at one of the vertices of the region. The following table gives *C* in thousands of dollars:

Vertex	x	y	$C(x,y) = 580x + 600y$
A	1,800	200	1,164
B	1,850	150	1,163
C	2,250	150	1,395
D	2,160	240	1,396.8

To minimize the cost of building the parking lot, there should be 1,850 public spaces and 150 employee spaces. However, the difference in the cost *s* for points A and B is only $1, so actually the selection of any point on the line segment from A to B would give a minimal cost. The cost will be about $1,163,000.

To maximize the weekly revenue, we need a weekly revenue function, *R*. Since we anticipate at least $1,000 per week per public space and $100 per week per employee space, that function is

$$R(x, y) = 1{,}000x + 100y$$

We apply the Corner Principle to this function, showing the weekly revenue in thousands of dollars:

Vertex	x	y	$R(x,y) = 1000x + 100y$
A	1,800	200	1,820
B	1,850	150	1,865
C	2,250	150	2,265
D	2,160	240	2,184

The weekly revenue will be maximized at $2,265,000 if the parking lot has 2,250 public spaces and 150 employee spaces.

The cost of making the parking lot may be the least at points A or B, but this is a one-time cost. However, the revenue is computed weekly. Thus, the proposal is to maximize the revenue by constructing 2,250 public spaces and 150 employee spaces.

Extension Questions:

- How can you investigate the cost of various combinations of public and employee parking spaces within the feasible region?

  You could make a table of the x- and y-coordinates of various points in the region and compute the corresponding costs.

- Would this be efficient to do?

  No. There are many points in the feasible region with integer coordinates.

- Instead of choosing points, what else might you choose?

  You could choose different values for the cost. Let C be the chosen cost and investigate the equation $580x + 600y = C$.

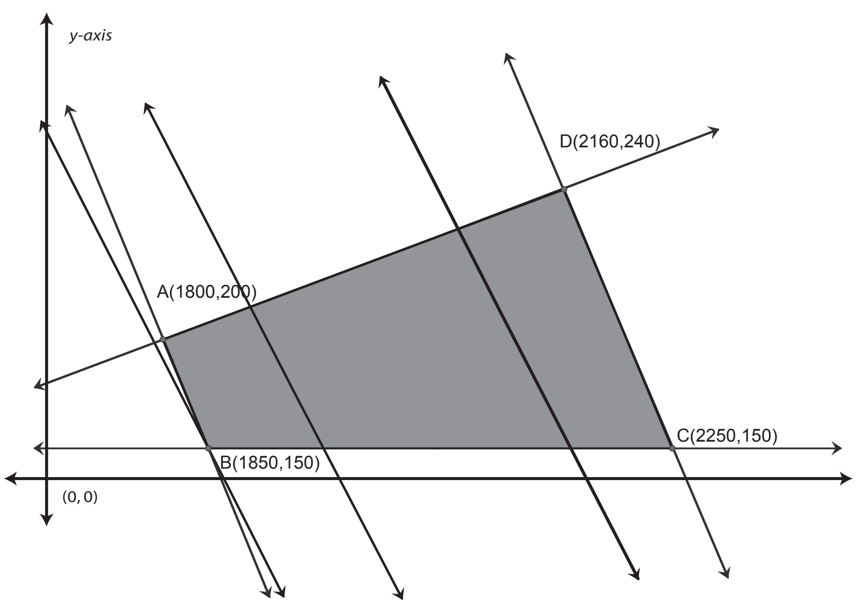
- How would you determine x- and y-values that will satisfy the equation $580x + 600y = C$?

  We could solve for y in terms of x to get $y = \frac{C - 580x}{600}$. *Then we could use the calculator table or graph.*

- Experiment with a few different C values close to the value that minimizes the cost. Graph the resulting cost equations. What do you notice about the graphs?

  The following have been graphed with the original functions:

$$y = \frac{1{,}163{,}000 - 580x}{600} \qquad y = \frac{1{,}192{,}000 - 580x}{600} \qquad y = \frac{1{,}308{,}000 - 580x}{600}$$



The graphs of the cost equations are parallel lines. When the cost is the minimum, $1,163,000, the line goes through (1,850,150).

- Can you make a conjecture about the location of the vertex that minimizes cost?

  Yes. Draw a line using a fixed cost. Move it parallel to itself from right to left across the feasible region. The last vertex it passes through will minimize the cost.

- Your calculator graph looks like the line corresponding to the minimal cost is the same as the boundary line $y = 2000 - x$. Is that true?

  No. The slope of the minimal cost equation is $m = \frac{-580}{600} = -0.9\overline{6}$*. That is so close to the slope of the boundary line, m = -1, it is hard to distinguish between the lines.*

- What would you conjecture to be true about the location of the vertex that will maximize the weekly revenue?

  Draw a line representing a particular weekly revenue, R, given by $1{,}000x + 10y = R$*. Move it parallel to itself from left to right across the feasible region. The last vertex the line passes through will maximize the revenue.*

The Mild and Wild Amusement Park

Three friends, Travis, Kaitlyn, and Karsyn, spent the day at Mild and Wild Amusement Park, which features rides classified as Mild, Wild, or Super Wild.

The park had 2 ticket purchase options.

> Option One: Pay a $5 admission fee and buy a ticket at regular price for each ride individually.
>
> Option Two: Pay a $5 admission fee and buy a ticket book that includes 8 tickets for each of the 3 different types of rides at a 20% discount per ticket.

The 3 friends chose to pay with Option One. They paid the admission fee plus the regular ticket cost for each ride they chose.

By the end of the day, Travis had ridden on 4 Mild rides, 8 Wild rides, and 8 Super Wild rides for a total ticket cost of $26. Kaitlyn had ridden on 8 Mild rides, 7 Wild rides, and 5 Super Wild rides for a total ticket cost of $24.25. Karsyn had ridden on 7 Mild rides, 6 Wild rides, and 4 Super Wild rides for a total ticket cost of $20.50.

1. Determine the ticket price for each type of ride—Mild, Wild, and Super Wild. Solve an algebraic system for this situation using matrices and technology.

2. Determine the amount each person would spend if he or she had chosen Option Two and ridden the same combination of rides. Explain which method of payment would have been best for each person for their day at the amusement park.

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus: (2A.3) Foundations for functions. The student formulates systems of equations and inequalities from problem situations, uses a variety of methods to solve them, and analyzes the solutions in terms of the situations.

The student is expected to:

(A) analyze situations and formulate systems of equations in two or more unknowns or inequalities in two unknowns to solve problems.

(B) use algebraic methods, graphs, tables, or matrices, to solve systems of equations or inequalities.

(C) interpret and determine the reasonableness of solutions to systems of equations or inequalities for given contexts.

Scaffolding Questions:

- What information is known in the problem?
- What are the unknowns in the problem?
- How will you organize the known information in a matrix?
- How will you then organize the unknowns and the total amount spent by the friends in matrices?
- What matrix equation can you now write?
- How do you solve the matrix equation?
- How will you use your calculator to solve the problem?

Sample Solutions:

1. We know the number of rides each person took and the total amount they paid for tickets. We need to determine the price of a ticket for each type of ride.

   We can write a matrix equation for this situation. The 3 by 3 coefficient matrix, *A*, will represent the 3 friends (rows) and the number of each of the 3 types of ride he or she took (columns). The ticket price matrix, *X*, will be a 3 by 1 matrix.

   Let m = the ticket price for a Mild ride

   w = the ticket price for a Wild ride

   s = the ticket price for a Super Wild ride.

   The constant matrix will be a 3 by 1 matrix, *P*, of the total each person paid for tickets.

$$A = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 7 & 5 \\ 7 & 6 & 4 \end{bmatrix}, \quad X = \begin{bmatrix} m \\ w \\ s \end{bmatrix}, \quad P = \begin{bmatrix} 26.00 \\ 24.25 \\ 20.50 \end{bmatrix}$$

The matrix equation to solve is $AX = P$. To do this we enter A and P as matrices in the calculator and get the solution by computing A^{-1} and the product $A^{-1}P$. This gives us the ticket price matrix X, since

$$A^{-1}AX = A^{-1}P$$
$$X = A^{-1}P$$
$$X = A^{-1}P$$
$$X = \begin{bmatrix} m \\ w \\ s \end{bmatrix} = \begin{bmatrix} 1.00 \\ 1.25 \\ 1.50 \end{bmatrix}$$

The regular ticket prices for Mild, Wild, and Super Wild rides are \$1, \$1.25, and \$1.50, respectively.

2. For Option Two, we need to know the discounted price of the tickets, which is 80% of the ticket price matrix, X.

$$D = 0.80X = 0.80\begin{bmatrix} 1.00 \\ 1.25 \\ 1.50 \end{bmatrix} = \begin{bmatrix} 0.80 \\ 1.00 \\ 1.20 \end{bmatrix}$$

For Option Two, each person would pay the admission fee and the price of the ticket book, which would be 5 + 8(0.80 + 1.00 +1.20) = 29 dollars.

The table below compares how the 3 friends would fare with each option.

	Spent with Option One: admission fee plus ticket cost	Cost of Option Two	Better buy
Travis	\$5.00 + \$26.00 = \$31.00	\$29.00	Option Two
Kaitlyn	\$5.00 + \$24.25 = \$29.25	\$29.00	Option Two
Karsyn	\$5.00 + \$20.50 = \$25.50	\$29.00	Option One

Travis and Kaitlyn would have gotten a better deal with Option Two, while Karsyn was better off with Option One.

Additional Algebra II TEKS: (2A.2) Foundations for functions. The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

The student is expected to:

(A) use tools including factoring and properties of exponents to simplify expressions and to transform and solve equations.

Connection to TAKS:

Objective 4: The student will formulate and use linear equations and inequalities.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

Extension Questions:

- What linear system corresponds to the matrix equation you solved in this problem? How can you obtain that system from the matrix equation?

  To get the corresponding linear system, we multiply each row in the coefficient matrix and the ticket price matrix, term by term, and sum the products. We set that equal to the corresponding entry in the constant matrix. The system is

$$\begin{aligned} 4m + 8w + 8s &= 26.00 \\ 8m + 7w + 5s &= 24.25 \\ 7m + 6w + 4s &= 20.50 \end{aligned}$$

- What method would you use to solve this system? How would the work to do that compare with the matrix solution?

  We would solve the system using the Linear Combination Method. This is not as efficient as solving the system as a matrix equation.

- Suppose the amusement park had a fourth type of ride, called Colossal Wild. In addition to the other rides, Travis rode 2 Colossal Wild rides and spent $30. Kaitlyn rode 3 Colossal Wild rides and spent $30.25. Karsyn rode 1 Colossal Wild ride and spent $22.50. Suppose we do not have the information found for the original problem. Would you be able to write and solve a matrix equation for this new situation?

  No. The coefficient matrix, A, would have 3 rows (one for each of the friends) and 4 columns (one for each type of ride). The ticket price matrix, X, would have 4 rows and 1 column. You cannot solve the equation AX = P by multiplying both sides of the equation by A^{-1}*. A is not a square matrix, so its inverse does not exist.*

- What would the corresponding linear system look like?

  It would be three equations in four unknowns, which we cannot solve.

  But we could figure out the ticket price for a Colossal Wild ride ticket if we knew the price of the others.

- In groups, create a situation involving four unknowns that will generate a 4 by 4 system. Groups will exchange situations, solve them, and share results.

  Answers will vary.

Task 1:

We found that the price of a mild ride was \$1, the price of a wild ride was \$1.25, and the price of a super wild ride was \$1.50. We determined this by setting up (1) algebraic system of equations, then entering the information into (2) 3 different matrices: a coeffecient matrix, a variable matrix, and an answer matrix.

(1)
$$4m + 8w + 8s = \$26.00$$
$$8m + 7w + 5s = \$24.25$$
$$7m + 6w + 4s = \$20.50$$

(2)
$$\begin{bmatrix} 4 & 8 & 8 \\ 8 & 7 & 5 \\ 7 & 6 & 4 \end{bmatrix} \begin{bmatrix} m \\ w \\ s \end{bmatrix} = \begin{bmatrix} \$26.00 \\ \$24.25 \\ \$20.50 \end{bmatrix}$$

We then set up this equation

(3)
$$\begin{bmatrix} M \\ W \\ S \end{bmatrix} = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 7 & 5 \\ 7 & 6 & 4 \end{bmatrix}^{-1} \begin{bmatrix} \$26.00 \\ \$24.25 \\ \$20.50 \end{bmatrix}$$

We solved this on the calculator and found M to be \$1, W to be \$1.25, and S to be \$1.50.

→

Task 2:

We needed to find out if the three friends would have saved money if they had chosen to pay with option **two**. We found this using the following methods.

First, we needed to determine the price of each ticket with a 20% discount. We did this by multiplying .2 by the cost of each ticket, then subtracting the product from the original price.

M	W	S
.2 x $1.00 = .20	.2 x $1.25 = .25	.2 x $1.50 = .30
$1.00 - .20 = **.80**	$1.25 - .25 = **$1.00**	$1.50 - .30 = **$1.20**

Then we needed to find the total cost of the package in option two. We found this by multiplpying 8 by $0.80, $1.00, and $1.50, then adding the products togeather. The sum was $24.00. We then added the $5 admission.

If they had bought option two, each person would have payed $29.

In buying option one, they each payed the entrance fee of $5.00 plus the total cost of their tickets.

Travis: $26.00 + $5.00 = $31.00
Kaitlyn: $24.25 + $5.00 = $29.25
Karsyn: $20.50 + $5.00 = $25.50

By subtracting the amount that they would have spent in buying option one from the amount that they spent by buying option two, we discovered that they would have saved the following amounts by buying option one.

Travis: $29.00 − $31.00 = $ 2.00
Kaitlyn: $29.00 − $29.25 = $−.25
Karsyn: $29.00 − $25.50 = $4.50

Therefore, Travis would have saved $2.00. Kaitlyn would have 25¢, and Karsyn would have lost $4.50.

Weather Woes

Storm E. Freeze has been fascinated with weather phenomena since childhood. After she graduates from college, Storm would like to become a meteorologist for a national weather syndicate, and she knows that she must be able to convert Celsius temperatures to Fahrenheit temperatures with ease. In addition, her job as a meteorologist will require that she be able to explain how the formulas for each are related and describe the conversions verbally, graphically, and symbolically.

She knows there is a linear relationship between the Celsius measure and the Fahrenheit measure. She has recorded the following measures:

Celsius temperature °*C*	**Fahrenheit temperature °*F***
5	41
14	57.2

You have agreed to help her with her math project involving temperature conversion and inverses.

1. Determine a formula to express *F* in terms of *C*.
2. Determine a formula to express *C* in terms of *F*.
3. Explain algebraically why these are inverse functions.
4. Graph the two functions on the same set of axes. Describe the graphs, their domains, and their ranges. How do the graphs help determine if the functions are inverses? Explain the meaning of the point of intersection of the two graphs.

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus: (2A.4) Algebra and geometry. The student connects algebraic and geometric representations of functions.

The student is expected to:

(A) identify and sketch graphs of parent functions, including linear ($f(x) = x$), quadratic ($f(x) = x^2$), exponential ($f(x) = a^x$), and logarithmic ($f(x) = \log_a x$) functions, absolute value of x ($f(x) = |x|$), square root of x ($f(x) = \sqrt{x}$), and reciprocal of x ($f(x) = 1/x$).

(B) extend parent functions with parameters such as a in $f(x) = a/x$ and describe the effects of the parameter changes on the graph of parent functions.

(C) describe and analyze the relationship between a function and its inverse.

Additional Algebra II TEKS:
None

Scaffolding Questions:

- What must you know to determine a linear function rule?
- What information can you get from the table?
- How can you determine whether two functions are inverses of one another?
- Graphs that are inverses of one another have a special property. What is that property?

Sample Solutions:

1. Use the table to determine the slope of the linear function.

Celsius temperature °*C*	Fahrenheit temperature °*F*
5	41
14	57.2

$$\frac{\text{change in } F}{\text{change in } C} = \frac{57.2 - 41}{14 - 9} = \frac{16.2}{9} = 1.8$$

$F = 1.8C + b$

Use one of the points to determine the value of b.

$41 = 1.8(5) + b$

$b = 32$

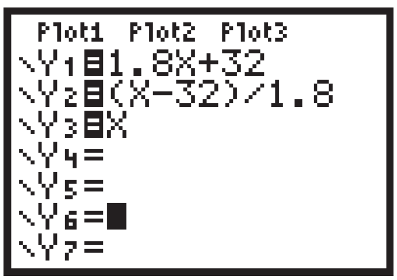
$F = 1.8C + 32$

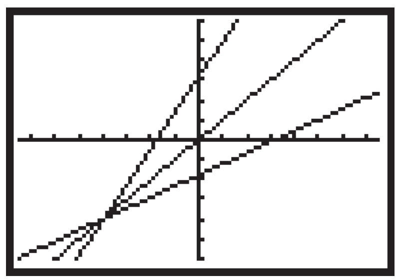
2. Rewrite the formula to solve for *C* in terms of *F*.

$$F = 1.8C + 32$$
$$F - 32 = 1.8C$$
$$C = \frac{F - 32}{1.8}$$

3. It makes sense that these formulas would be inverses of each other if, when you convert from one temperature scale to another and back again, you arrive at the original temperature.

4. Enter the graphs of the two functions into a graphing calculator. The line $y = x$ is also graphed to show the relationship between the functions.





Both functions are linear functions. For every point (*a, b*) on the graph of the original function, you can find the point (*b, a*) on the graph of the inverse function. The graphs of the function and its inverse are symmetric to each other with respect to the line $y = x$. Graphs of the function and its inverse are reflections of each other over the line $y = x$.

This relationship can also be shown algebraicially.

$$C = 1.8F + 32$$
$$F = \frac{C - 32}{1.8}$$

Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

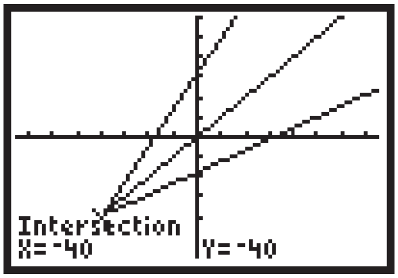
We can check to see if the functions are inverses by substituting for *F* in the first rule.

$$C = 1.8\left(\frac{C-32}{1.8}\right)+32$$
$$C = C-32+32 = C$$

The functions are inverses of each other.

The horizontal axis and the vertical axis represent degree measures. The domain of one of the functions is equal to the range of its inverse. The range of the function is equal to the domain of the inverse. Each number in the domain corresponds to a unique number in its range and vice versa.

The meaning of the intersection point is that if a measure is -40 degrees in Celsius it is equal to -40 degrees in Fahrenheit.



Extension Questions:

- The functions modeled in the problem were from only one family of functions. Consider all of the other families of functions,

  $$y = x, y = x^2, y = |x|, y = \sqrt{x}, y = a^x, y = \log_a x, \text{ and } y = \frac{1}{x}$$

  and determine any functions that are inverses of each other.

  An inverse function has the characteristic of "undoing" the operations of the original function. The inverse of the function y = x^2 is the square root function $y = \sqrt{x}$ if you restrict the domain and range of the function to be all non-negative real numbers.

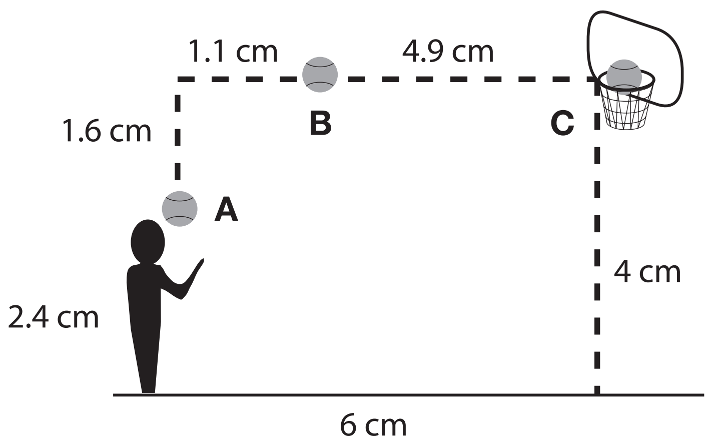
  The inverse of a power function such as y = 2^x is y = log2^x.

  The function y = x is its own inverse. The function $y = \frac{1}{x}$ is its own inverse.

Chapter 4:
Quadratic Functions

Basketball Throw

A student, Foe Tagrafer, took yearbook pictures at the district championship basketball game. While Ray Bounder was taking the game-winning free throw, Foe took three pictures. The ball was at positions A, B, and C on the diagram below when the pictures were taken. Foe knew that the distance from Ray's feet to the base of the goal was 15 feet. He created a scale diagram from the three photographs.



1. Using the scale diagram, determine the actual measurements for each situation.

Picture	Horizontal Distance before (-) or after (+) the front of the goal	Vertical Distance above (+) or below (-) the goal
A		
B		
C		

2. Find a quadratic function to model the relationship between these two quantities:
   - Horizontal distance before or after the goal
   - Vertical distance above or below the goal

3. Graph the function using a graphing calculator. What windows did you use? Justify your choice.

4. For these values, find how high off the ground the ball got at its maximum height.

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus: (2A.6) Quadratic and square root functions. The student understands that quadratic functions can be represented in different ways and translates among their various representations.

The student is expected to:

(A) determine the reasonable domain and range values of quadratic functions, as well as interpret and determine the reasonableness of solutions to quadratic equations and inequalities.

(B) relate representations of quadratic functions, such as algebraic, tabular, graphical, and verbal descriptions.

Scaffolding Questions:

- Describe how to find actual measurements from the scale diagram's measurements.
- How would you position a coordinate system on the diagram?
- What kind of function would describe the trace of the ball in the diagram?
- If your function does not measure height of the ball from the ground, how can you use your function to determine this height?

Sample Solutions:

1. Because the diagram is drawn to scale, the ratio of the distances in the diagram in centimeters to the actual distances in feet is 6 cm to 15 feet.

   Point A:

   The horizontal distance is given to be 15 feet. Because Ray is positioned before the goal, the distance is represented as -15.

   Vertical distance on the diagram is 1.6 cm. Use the relationship between 6 and 15 to solve for *a*, the actual distance.

$$\frac{6}{15} = \frac{1.6}{a}$$
$$a = 4$$

   The actual distance is 4 feet below the goal. It is represented as -4.

   Point B:

   The vertical distance (above or below the goal) is 0.

   The horizontal distance is 4.9 cm on the diagram. Determine the actual distance using the ratio of the diagram distance of 6 cm to the actual distance of 15 feet.

$$\frac{6}{15} = \frac{4.9}{b}$$
$$b = 12.25$$

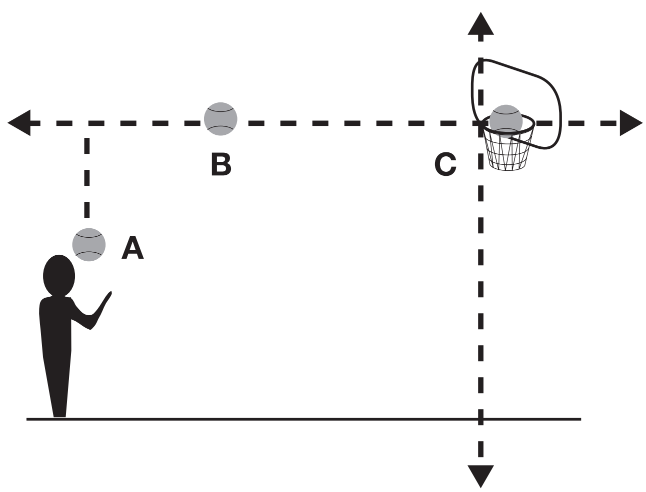
The distance is 12.25 feet before the goal. It is represented as -12.25.

Point C:

Point C is at the goal so both horizontal and vertical distances are 0 feet.

Picture	Horizontal Distance before (-) or after (+) the front of the goal	Vertical Distance above (+) or below (-) the goal
A	-15	-4
B	-12.25	0
C	0	0

2. Position the axes as shown in the picture.



The parabola will pass through three points: (-15, -4), (-12.25, 0), and (0, 0).

(2A.8) Quadratic and square root functions. The student formulates equations and inequalities based on quadratic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

(A) analyze situations involving quadratic functions and formulate quadratic equations or inequalities to solve problems.

(D) solve quadratic equations and inequalities using graphs, tables, and algebraic methods.

Notes

Additional Algebra II TEKS: (2A.3) Foundations for functions. The student formulates systems of equations and inequalities from problem situations, uses a variety of methods to solve them, and analyzes the solutions in terms of the situations.

The student is expected to:

(A) analyze situations and formulate systems of equations in two or more unknowns or inequalities in two unknowns to solve problems.

(B) use algebraic methods, graphs, tables, or matrices, to solve systems of equations or inequalities.

(C) interpret and determine the reasonableness of solutions to systems of equations or inequalities for given contexts.

The function rule for the parabola is of the form $y = ax^2 + bx + c$.

The point (0, 0) is on the graph. Thus, when $x = 0$, $y = 0$.

$$0 = a(0)^2 + b(0) + c$$

$$c = 0$$

Use the other two points to get a system of equations.

$$-4 = a(-15)^2 + b(-15)$$
$$0 = a(-12.25)^2 + b(-12.25)$$

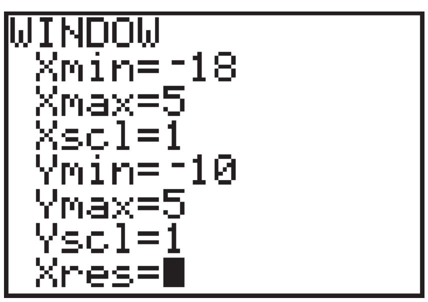
Simplify the two equations and solve for a and b.

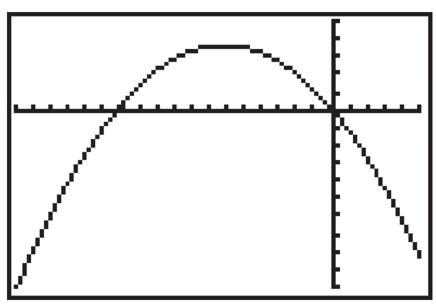
$-4 = 225a - 15b$ and $0 = 150.0625a - 12.25b$

$$b = \frac{150.0625}{12.25}a$$
$$b = 12.25a$$
$$-4 = 225a - 15(12.25a)$$
$$-4 = 225a - 183.75a$$
$$-4 = 41.25a$$
$$a = -0.097$$
$$b = 12.25a = 12.25(-0.097) = -1.188$$

The quadratic function is $y = -0.097x^2 - 1.188x$.

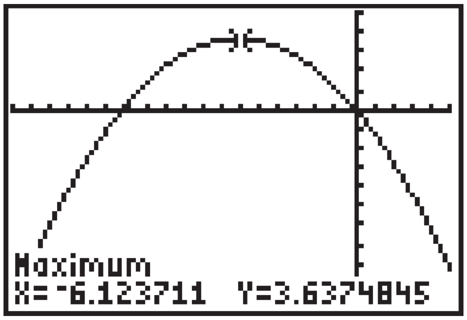
3. The graphing window was selected based on the values in the table. The range of values for x is from -15 to 0 and the range of values for y is from -4 to about 5.





4. The high point may be obtained by finding the maximum point on the graph. The maximum *y*-value occurs at *x* = -6.123 or when the horizontal distance of the ball from the goal is 6.123 feet. The *y*-value of this point must be added to the height of the basketball to determine the distance from the ground.

   Hence the ball is 3.637 + 10, or 13.637 feet above the ground at its maximum height.



Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

Extension Questions:

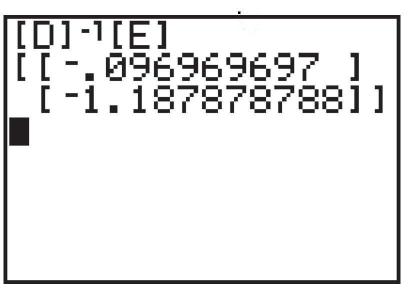
- Describe another approach to solving the system of equations used to determine the values of *a* and *b*.

  The system could be solved using matrices.

$$D = \begin{bmatrix} (-15)^2 & -15 \\ (-12.25)^2 & -12.25 \end{bmatrix} \qquad E = \begin{bmatrix} -4 \\ 0 \end{bmatrix} \qquad X = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$DX = E$$

$$X = D^{-1}E$$



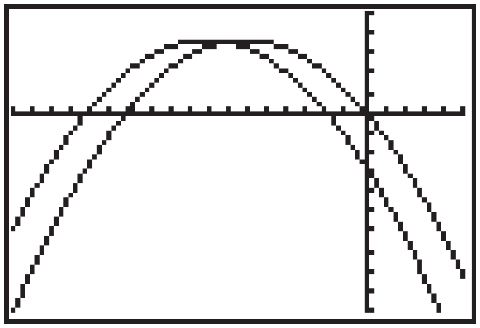
a = -0.097 and *b* = -1.188

- If Ray Bounder had stepped 2 feet back (away from the goal) and thrown the ball in exactly the same manner, would the ball land in the basket?

No. If he steps 2 feet back, the graph that represents the relationship between horizontal and vertical distances is shifted horizontally 2 units to the left. The function rule is

$$y = -0.097(x + 2)^2 - 1.188(x + 2)$$

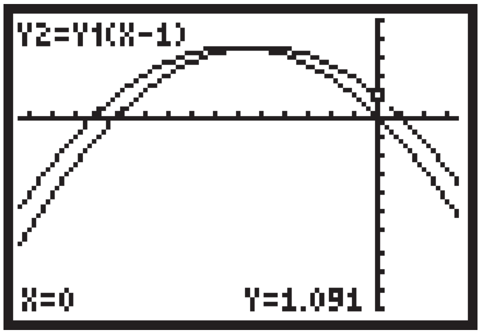
The new graph and the original graph are shown below. The ball would travel below the goal.



- What if Ray had stepped 1 foot closer to the basket and thrown the ball in the same manner?

The equation is $y = -0.097(x - 1)^2 - 1.188(x - 1)$.

It appears that the ball would be 1.091 feet above the goal.



Fixed Perimeter Rectangles

You have a flexible fence of length $L = 130$ meters. You want to use all of this fence to enclose a rectangular plot of land of at least 800 square meters in area.

1. Determine a function for the area of the plot of land in terms of either the width or the length.
2. Sketch a graph of the function and determine a reasonable domain and range.
3. Illustrate on the graph the values for which the area of the plot of land is at least 800 meters.
4. Solve problem 3 algebraically.
5. Describe the relationship between the graphical and algebraic solutions.
6. What dimensions will give a plot of land with the greatest area?

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus:
(2A.6) Quadratic and square root functions. The student understands that quadratic functions can be represented in different ways and translates among their various representations.

The student is expected to:

(A) determine the reasonable domain and range values of quadratic functions, as well as interpret and determine the reasonableness of solutions to quadratic equations and inequalities.

(B) relate representations of quadratic functions, such as algebraic, tabular, graphical, and verbal descriptions.

Additional Algebra II TEKS:
(2A.8) Quadratic and square root functions. The student formulates equations and inequalities based on quadratic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

(A) analyze situations involving quadratic functions and formulate quadratic equations

Scaffolding Questions:

- Draw a diagram of some possible rectangular plots with a perimeter of 130 meters.
- If the length of one side of the plot is 40 meters, and the perimeter is 130 meters, draw the plot.

  What is the length of the other side?

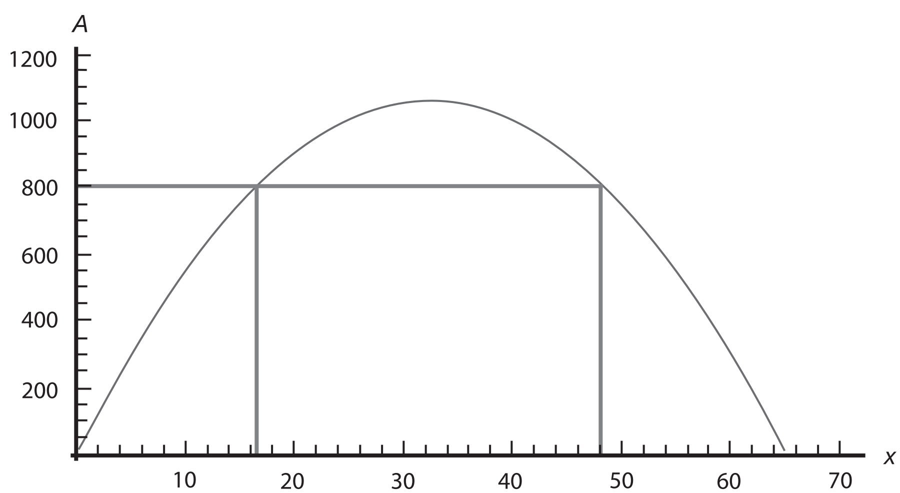
  What is the area of the plot?
- If the length of one side of the plot is x meters, and the perimeter is 130 meters, what is the length of the other side? What is the area?
- Graph the area of the plot in terms of x .

Sample Solutions:

1. Since the perimeter is 130 meters, half the perimeter is 65 meters. If the length of one side of the plot is x meters, the length of the other side is $65 – x$ meters, and the area is $x\,(65 – x)$ square meters:

$$A(x) = x\,(65 – x)$$

2. Here is a graph of $A(x)$. It is an inverted parabola:



The domain for this function is $0 < x < 65$,
and the range is $0 < y < 1056.25$.

3. The auxiliary lines drawn show that if the side length x is between about 17 meters and about 48 meters, the area will be 800 square meters or more.

4. To find these dimensions exactly, solve the following equation for x:

$$A(x) = x\,(65 - x) = 800$$

This is a quadratic equation in x, and we can write it as

$$x^2 - 65x + 800 = 0$$

Its solutions can be found by the quadratic formula:

$$x = \frac{65}{2} \pm \frac{1}{2}\sqrt{65^2 - 4(800)} - \approx 32.5 \pm 16$$

The solutions are approximately x = 16.5 meters and x = 48.5 meters. Since half the perimeter is 65 meters, the side length 16.5 meters of one side of the rectangle corresponds to a side length of 65 – 16.5 = 48.5 meters for the other side. Therefore both solutions correspond to a 16.5 by 48.5 rectangle.

5. To ask when the area is greater than or equal to 800 square meters, solve the equation $x(65 - x) = 800$ and determine the values of x for which the area is equal to 800 square meters. On the graph these values are the intersection of the lines $y = 800$ and the parabola $y = x(65 - x)$.

6. Since the parabola that is the graph of $A(x)$ is symmetric about its two zeros $x = 0$ and $x = 65$, the largest value of the area will occur halfway in between these, at $x = 32.5$. The other side length is then 65 – 32.5 = 32.5 meters. In other words, the rectangle giving the largest area is a square. The area of this square is $32.5^2 = 1{,}056.25$ square meters.

or inequalities to solve problems.

(B) analyze and interpret the solutions of quadratic equations using discriminants and solve quadratic equations using the quadratic formula.

(C) compare and translate between algebraic and graphical solutions of quadratic equations.

(D) solve quadratic equations and inequalities using graphs, tables, and algebraic methods.

Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

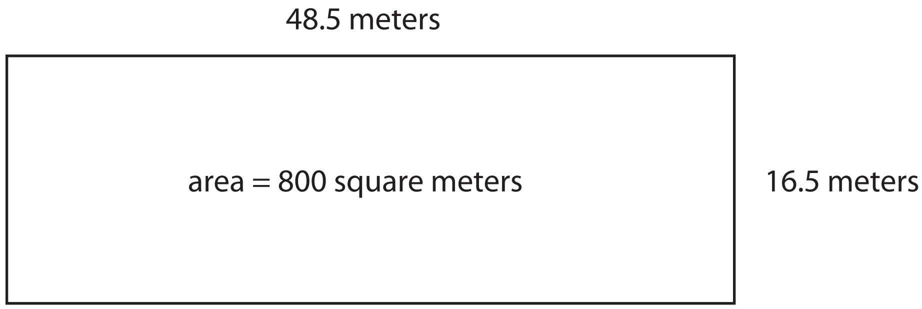
Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

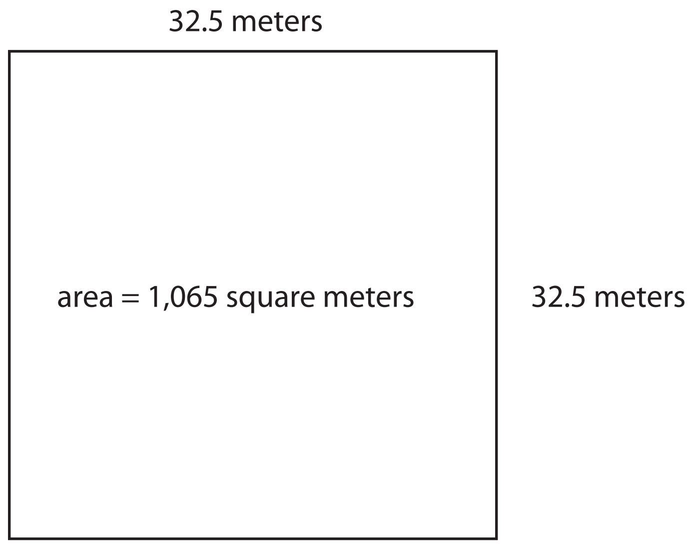
Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

Here are diagrams showing the range of shapes of plots with perimeter 130 meters and area at least 800 square meters.





Extension Questions:

- Solve the problem generally. If you have a fence of fixed length P_0 meters, and want the area of a rectangular plot enclosed by this fence to be at least A_0 square meters, what are the possible dimensions of the plot?

  Since the perimeter is P_0 meters, half the perimeter is $\frac{P_0}{2}$ meters. If the length of one side of the plot is x meters, the length of the other side is $\frac{P_0}{2} - x$ meters, and the area is $x\left(\frac{P_0}{2} - x\right)$ square meters:

  $$A(x) = x\left(\frac{P_0}{2} - x\right)$$

  The requirement that the area be at least A_0 square meters leads to the inequality:

  $$x\left(\frac{P_0}{2} - x\right) \geq A_0$$

  This is a quadratic inequality in x, and we can write it as

  $$x^2 - \frac{P_0}{2}x + A_0 \leq 0$$

  Its solutions can be found by first solving the corresponding quadratic equation

  $x^2 - \frac{P_0}{2}x + A_0 = 0$ *using the quadratic formula. Its solutions are*

  $$x = \frac{-\frac{P_0}{2} + \sqrt{\left(\frac{P_0}{2}\right)^2 - 4A_0}}{2} \text{ or } \frac{-\frac{P_0}{2} - \sqrt{\left(\frac{P_0}{2}\right)^2 - 4A_0}}{2}$$

  Therefore the smallest one side can be and still yield at least an area of A_0 is

  $$\frac{-\frac{P_0}{2} - \sqrt{\left(\frac{P_0}{2}\right)^2 - 4A_0}}{2} \text{ or } -\frac{P_0}{4} - \frac{1}{2}\sqrt{\left(\frac{P_0}{2}\right)^2 - 4A_0}$$

- What are the largest and smallest areas that can be enclosed in a rectangular plot with a fixed perimeter P_0?

  There is no smallest area, since by making the rectangle very long and thin, the area can be made as small as desired and still have perimeter P_0.

The largest area will occur when the rectangle is a square with sides $\frac{P_0}{4}$*. Its area is* $\left(\frac{P_0}{4}\right)^2$*.*

Motion Under Gravity

If a ball is thrown straight upward from ground level with a velocity equal to $v_0 \frac{\text{meters}}{\text{sec}}$, physics tells us that its height *h(t)* after time *t* is given by

$$h(t) = v_0 t - \frac{1}{2} g t^2$$

Here, *g* is the gravitational acceleration.

Near the earth's surface *g* has a value of about 9.8 $\frac{\text{meters}}{\text{sec}^2}$.

Near the moon's surface *g* has a value of about 1.6 $\frac{\text{meters}}{\text{sec}^2}$.

Suppose a ball is thrown upward from the earth's surface with a velocity of v_0 = 25 meters per second. (This is about as fast as a person could throw a ball upward.)

1. Determine the maximum height the ball reaches if thrown on Earth. What would be the maximum height the ball reaches if it were to be launched upward with the same velocity on the moon? In each situation show the relationship between the time when the maximum occurs and the roots of the function.

2. Determine the ratio of maximum height on the moon to the maximum height on the earth, for the given launch velocity. Find this ratio using at least two other launch velocities. What do you conjecture about this ratio?

3. If you wanted the ball on the moon to stay aloft for the *same length of time* as the ball on the earth when it is thrown at 25 meters per second, what should the launching velocity be on the moon?

4. If you wanted the ball on the moon to reach the *same maximum height* as the ball on the earth when it is thrown at 25 meters per second, what should the launching velocity be on the moon?

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus: (2A.6) Quadratic and square root functions. The student understands that quadratic functions can be represented in different ways and translates among their various representations.

The student is expected to:

(A) determine the reasonable domain and range values of quadratic functions, as well as interpret and determine the reasonableness of solutions to quadratic equations and inequalities.

(B) relate representations of quadratic functions, such as algebraic, tabular, graphical, and verbal descriptions.

.**Scaffolding Questions:**

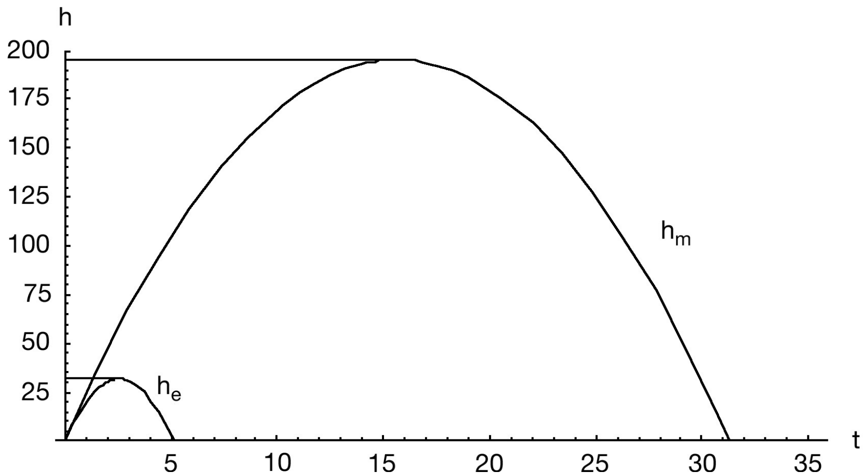
- For different gravity for the earth and the moon, how do you expect the trace of the ball to be different if the initial velocity is the same?
- Sketch a graph of the height function $h(t) = v_0 t - \frac{1}{2} g t^2$ for the value v_0 = 25 meters per second and $g = 9.8 \frac{\text{meters}}{\text{sec}^2}$. For what value(s) of the time *t* does this function have the value 0?
- Sketch a graph of the height function $h(t) = v_0 t - \frac{1}{2} g t^2$ for the value v_0 = 25 meters per second and $g = 1.6 \frac{\text{meters}}{\text{sec}^2}$. For what value(s) of the time *t* does this function have the value 0?
- If a ball is thrown upward from the earth's surface with a velocity of v_0 = 25 meters per second, for what value of the time *t* is the height the greatest? What is that height?
- If a ball is thrown upward from the moon's surface with a velocity of v_0 = 25 meters per second, for what value of the time *t* is the height the greatest? What is that height?
- If the ball must stay in the air the same length of time on the moon as it did when thrown on the earth, what equation must you solve?
- If you know the maximum height when the ball is thrown on the earth, how can you use that value to determine the time it reaches that same maximum height when it is thrown on the moon?

Sample Solutions:

1. Using what we are told about the physics, we see that the height of the ball above the surface at a time t seconds after launching is given by these functions:

Earth: $h_e(t) = 25t - 4.9t^2$ Moon: $h_m(t) = 25t - 0.8t^2$

Here are graphs of each:



The maximum height the ball reaches can be found from the graphs: It is about 30 meters on the earth, and about 190 meters on the moon.

To find these values more exactly, we can find in each case for what value of the time t the height is 0, and then use the symmetry of the parabolic graph of the function to evaluate the height at half this value of *t:*

Earth:
$$25t - 4.9t^2 = 0$$
$$t(25 - 4.9t) = 0$$
$$t = 0 \text{ or } t = \frac{25}{4.9}$$

The ball will reach its maximum height at the vertex of the parabola. Because of symmetry this will occur halfway between the two points where the height is 0.

Additional Algebra II TEKS:

(2A.8) Quadratic and square root functions. The student formulates equations and inequalities based on quadratic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

(A) analyze situations involving quadratic functions and formulate quadratic equations or inequalities to solve problems.

(C) compare and translate between algebraic and graphical solutions of quadratic equations.

(D) solve quadratic equations and inequalities using graphs, tables, and algebraic methods.

Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

At $t = \frac{1}{2} \bullet \frac{25}{4.9} \bullet = \frac{25}{9.8}$

The height at this time is $h_e(\frac{25}{9.8}) = 25(\frac{25}{9.8}) - 4.9(\frac{25}{9.8})^2 \approx 31.9$ meters .

Moon:

$$25t - 0.8t^2 = 0$$
$$t(25 - 0.8t) = 0$$
$$t = 0 \text{ or } t = \frac{25}{0.8}$$

The midpoint is at $t = \frac{1}{2} \bullet \frac{25}{0.8} = \frac{25}{1.6}$.

The maximum height is $h_m(\frac{25}{1.6}) = 25(\frac{25}{1.6}) - 0.8(\frac{25}{1.6})^2 \approx 195$ meters

2. The ratio of the maximum height on the moon to the maximum height on Earth for the given problem is $\frac{195}{31.9} \cong 6.11$.

To generate the rules for other velocities, we can use the general height function given in the problem

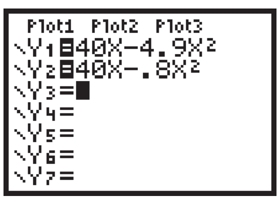
$$h(t) = v_0 t - \frac{1}{2} g t^2$$

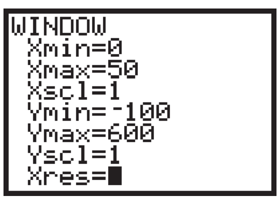
Let the velocity be 40 meters per second.

The two equations are

Earth: $h(t) = 40t - 4.9t^2$ Moon: $h(t) = 40t - 0.8t^2$

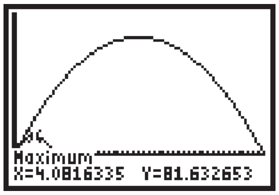
The question may be answered by graphing the two functions and finding their maximum height using a calculator.





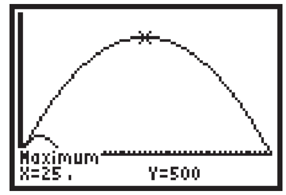
$h(t) = 40t - 4.9t^2$

The maximum height is 81.632 meters.



$h(t) = 40t - 0.8t^2$

The maximum height is 500 meters.



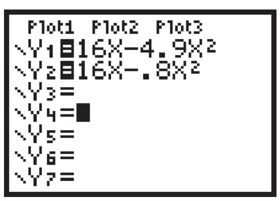
The ratio of the maximum height on the moon to the maximum height on Earth is $\frac{500}{81.63} \cong 6.12$.

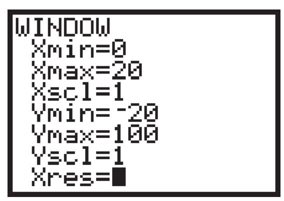
Let the velocity be 16 meters per second.

The two equations are

Earth: $h(t) = 16t - 4.9t^2$ Moon: $h(t) = 16t - 0.8t^2$

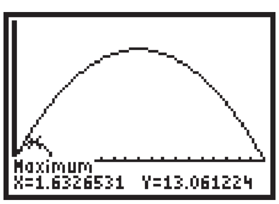
Graph the two functions:





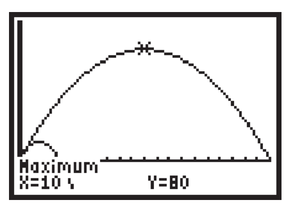
$h(t) = 16t - 4.9t^2$

The maximum height is 13.061 meters.



$h(t) = 16t - 0.8t^2$

The maximum height is 80 meters.



The ratio of the maximum height on the moon to the maximum height on Earth is $\frac{80}{13.061} \cong 6.125$.

A conjecture is that the ratio of the maximum height on the moon to the maximum height on the earth for a given velocity appears to be the same no matter what launch velocity is used.

3. Since the ball starts at ground level, the time the ball is in the air on the earth is the time when the maximum height is again zero. The value that was found in problem 1 is $t = \frac{25}{4.9}$. The rule for the height on the moon is $h(t) = v_m\, t - 0.8\, t^2$, where v_m represents the launching velocity. Substitute $t = \frac{25}{4.9}$ and *h(t)=0* in this rule.

$$0 = v_0 \frac{25}{4.9} - 0.8\left(\frac{25}{4.9}\right)^2$$
$$v_0 \frac{25}{4.9} = 0.8\left(\frac{25}{4.9}\right)^2$$
$$v_0 \approx 4.08$$

The velocity on the moon should be about 4.08 meters per second.

4. The maximum height of the ball on the earth is 31.9 meters. First find the time in terms of v_m. We know that the value of *t* for which this occurs on the moon is the midpoint of the two roots of the function.

$$h(t) = v_m\, t - 0.8\, t^2$$
$$0 = v_m\, t - 0.8\, t^2$$
$$0 = t(v_m - 0.8\, t)$$
$$t = 0 \quad \text{or} \quad t = \frac{v_m}{0.8}$$

The midpoint occurs at $t = \frac{1}{2} \bullet \frac{v_m}{0.8} = \frac{v_m}{1.6}$.

We want to know the value of the velocity that will make the maximum height equal to 31.9 meters.

$$31.9 = v_m\, t - 0.8\, t^2$$
$$31.9 = v_m\left(\frac{v_m}{1.6}\right) - 0.8\left(\frac{v_m}{1.6}\right)^2$$
$$31.9 = \frac{v_m^{\ 2}}{1.6} - 0.8\frac{v_m^{\ 2}}{1.6^2}$$
$$v_m^{\ 2} = 102.08$$
$$v_m \approx 10.1$$

The ball should be thrown on the moon at about 10.1 meters per second.

Extension Questions:

- Prove your conjecture from problem 2.

  Determine when the height is 0.

$$0 = v_0 t - \frac{1}{2}gt^2$$
$$0 = t\left(v_0 - \frac{1}{2}gt\right)$$
$$t = 0 \text{ or } t = \frac{2v_0}{g}$$

The height will be 0 when $t = \frac{2\ v_0}{g}$, *so the height will be a maximum at half of this value or* $t = \frac{1}{2}\frac{2v_0}{g} = \frac{v_0}{g}$ *. This represents the time when the ball will be at its maximum height.*

The maximum height will be

$$h\left(\frac{v_0}{g}\right) = v_0\left(\frac{v_0}{g}\right) - \frac{1}{2}g\left(\frac{v_0}{g}\right)^2 = \frac{v_0^{\ 2}}{2g}$$

The ratio of the maximum heights on the moon and on the earth, for a given launch velocity v_0*, is*

$$\frac{\text{maximum height on moon}}{\text{maximum height on earth}} = \frac{\frac{v_0^{\ 2}}{2g_m}}{\frac{v_0^{\ 2}}{2g_e}} = \frac{v_0^{\ 2}}{2g_m}\ \frac{2g_e}{v_0^{\ 2}} = \frac{g_e}{g_m} \approx \frac{9.8}{1.6} \approx 6.1.$$

This shows that for a given launch velocity, the ratio of the maximum height on the moon to the maximum height on Earth is about 6.1. That is, an object goes about 6 times as high on the moon as on the earth. Note that this ratio is independent of the velocity, v_0*.*

- Assume that a ball is launched from the ground with an initial velocity of 25 $\frac{\text{meters}}{\text{sec}}$ at an angle of 60° to the ground. Plot the actual path of the ball, and find its maximum height and where it lands.

To determine the horizontal and vertical velocities use right triangle trigonometry.

The vertical velocity is $\frac{1}{2} \bullet 25 \bullet \sqrt{3} \approx 21.7 \text{ m/s}$.

The horizontal velocity is 12.5 m/s.

If the ball is thrown from ground level, then the initial position is vertical height is 0 meters.

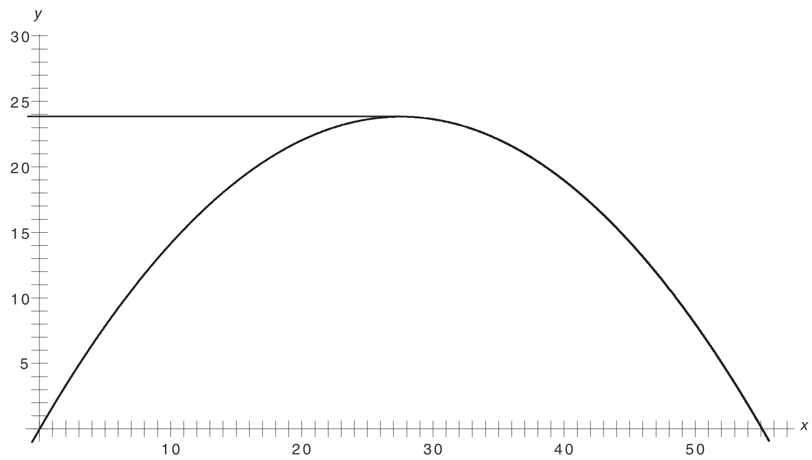
We know the vertical position is given by $y = 21.7\,t - \frac{1}{2}(9.8)\,t^2$.

The horizontal position is given by x = 12.5 t.

Solving for t gives $t = \frac{x}{12.5}$. *Putting this value of t into h(t) gives*

$$y = 21.7\left(\frac{x}{12.5}\right) - \frac{1}{2}(9.8)\left(\frac{x}{12.5}\right)^2 \approx 1.73\,x - 0.0314\,x^2$$

Here is a graph:



From the graph it can be determined that the maximum height is about 24 meters and the range about 57.5 meters.

- Now look at a case where a ball is launched from the earth with velocity 25 $\frac{\text{meters}}{\text{sec}}$ at an angle of 30° to the ground. Plot the actual path of the ball, and find its maximum height and where it lands.

The components are reversed. The vertical position is given by

$$y = h(t) = 12.5\ t - \frac{1}{2}(9.8)\ t^2.$$

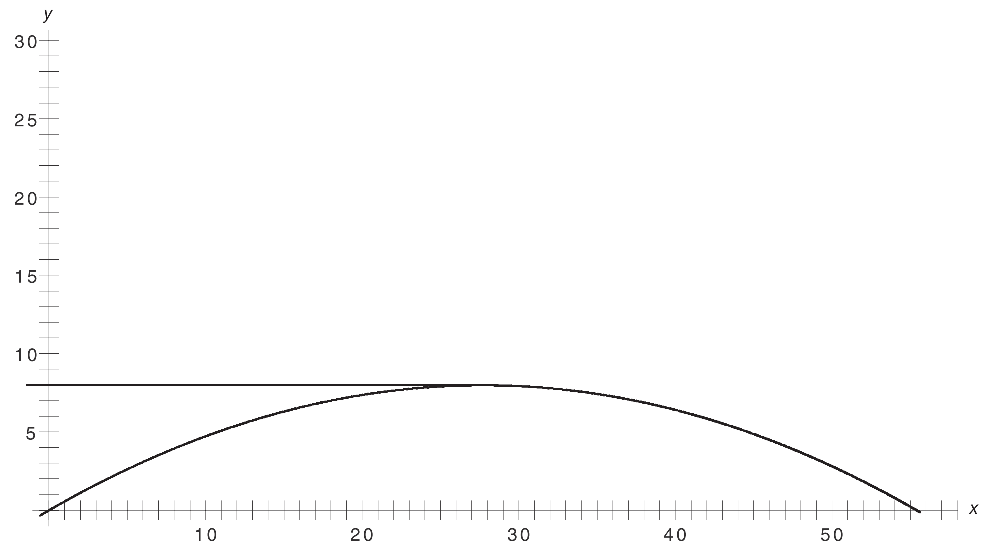
The horizontal position is given by

$$x = x(t) = 21.7\ t$$

Solving for t gives $t = \frac{x}{21.7}$ *. Putting this value of t into h(t) gives*

$$y = h(t) = 12.5\left(\frac{x}{21.7}\right) - \frac{1}{2}(9.8)\left(\frac{x}{21.7}\right)^2 \approx 0.576\ x - 0.0104\ x^2$$

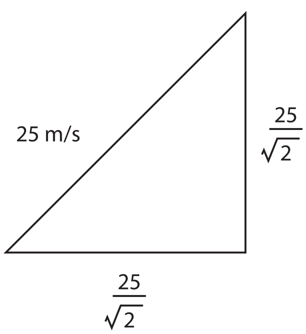
Here is a graph:



The maximum height is about 8 meters and the range about 55 meters.

- Now look at a case where a ball is launched from the earth with velocity 25 m/s at an angle of 45° to the ground. Plot the actual path of the ball, and find its maximum height and where it lands.

Consider the right triangle with a 45-degree angle and a hypoteneuse of 25 meters.

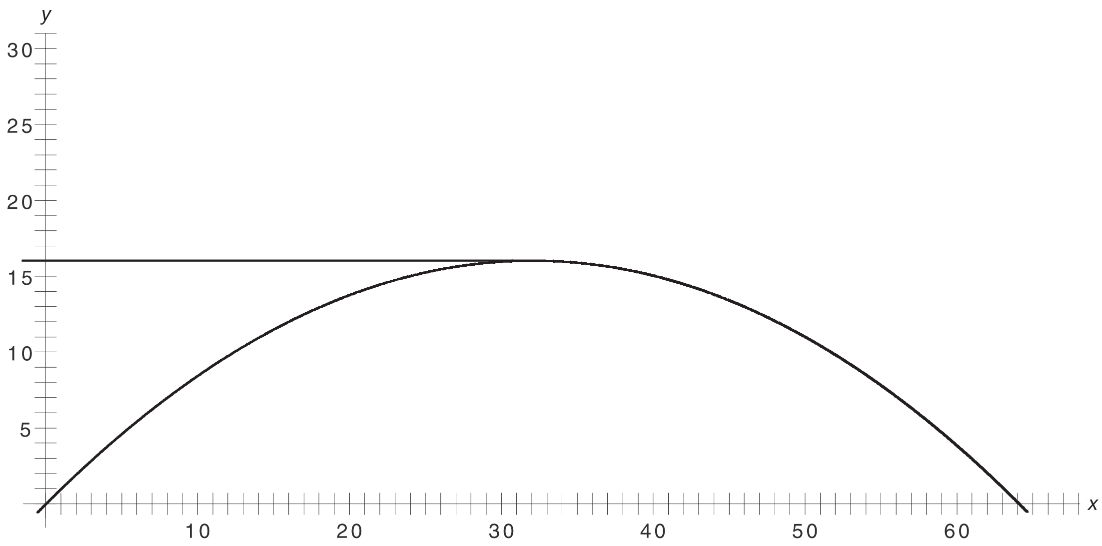


The two legs of the triangle are $\frac{25}{\sqrt{2}} \approx 17.7$ *meters per second.*

The vertical position is given by $y = 17.7\, t - \frac{1}{2}(9.8)\, t^2$.
The horizontal position is given by $x = 17.7\, t$.
Solving for t gives $t = \frac{x}{17.7}$. *Putting this value of t into h(t) gives*

$$y = x - \frac{1}{2}(9.8)\left(\frac{x}{17.7}\right)^2 \approx x - 0.0156\, x^2$$

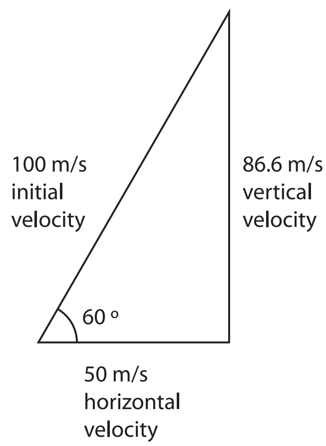
Here is a graph:



The maximum height is about 16 meters and the range about 64 meters.

Parabolic Paths

If a ball is launched upward at an angle, the horizontal and vertical motions can be tracked separately. For example, a ball launched upward at an angle of 60° to the ground at 100 meters per second (m/s), as shown in the diagram below, will have vertical velocity of about 86.6 meters per second and horizontal velocity of 50 meters per second.



1. The laws of physics tell us that its vertical height *y* after time *t* in seconds with an initial vertical velocity of *v* is given by

$$y = h_0 + vt - \frac{1}{2} g t^2$$

   Here, *g* is the gravitational acceleration, which near the earth's surface has a value of about 9.8 $\frac{\text{meters}}{\text{sec}^2}$, and h_0 is the initial height of the ball from the ground.

   A ball is launched upward from the top of a 75-meter tower with initial velocity 100 meters per second at an angle of 60° to the ground. Write the function for the vertical height as measured from the launch point, *y*, in terms of time *t*.

2. The horizontal distance from the launch point is given by the formula $x = wt$, where *w* meters per second is the horizontal velocity, *x* is distance in meters, and *t* is time in seconds. Write an expression for the horizontal distance for the situation described above.

3. Use the functions for both vertical height and horizontal distance as a function of time *t* to write the vertical height as a function of the horizontal distance.

4. Create a graph showing the actual path of the ball from where it is launched to where it hits the ground. Then use the graph to find its maximum height and where it hits the ground.

5. Suppose that the ball had been launched upward from the top of the 75-meter tower with an initial velocity of 80 meters per second at an angle of 60° to the ground. How would the diagram above be different from the diagram for the original situation? What are the new functions for vertical height and horizontal distance as a function of time *t*?

6. Determine the ball's maximum height and where it hits the ground.

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus:

(2A.6) Quadratic and square root functions. The student understands that quadratic functions can be represented in different ways and translates among their various representations.

The student is expected to:

(A) determine the reasonable domain and range values of quadratic functions, as well as interpret and determine the reasonableness of solutions to quadratic equations and inequalities.

(B) relate representations of quadratic functions, such as algebraic, tabular, graphical, and verbal descriptions.

Scaffolding Questions:

- What are the initial vertical and horizontal velocities?
- What is the initial vertical position?
- What is the initial horizontal position?
- What is the vertical height when the ball hits the ground?
- How much time will it take for the ball to hit the ground?
- Describe how to find t in terms of x.
- Describe how to find y in terms of x.
- Describe the graph. Where does the maximum height occur?
- Examine the graph and tell what it means to say that the ball has hit the ground. What is the height at that point?
- Draw the triangle for the initial velocity of 80 m/s with a 60° angle. Describe the relationship with the original triangle.

Sample Solutions:

1. We can assign the initial position of the ball in an x-y coordinate system as $x_0 = 0$, $y_0 = 75$. The initial vertical velocity, v_0 , is given to be 86.6 meters per second.

   From the given height function we know that the vertical position as a function of time is given by

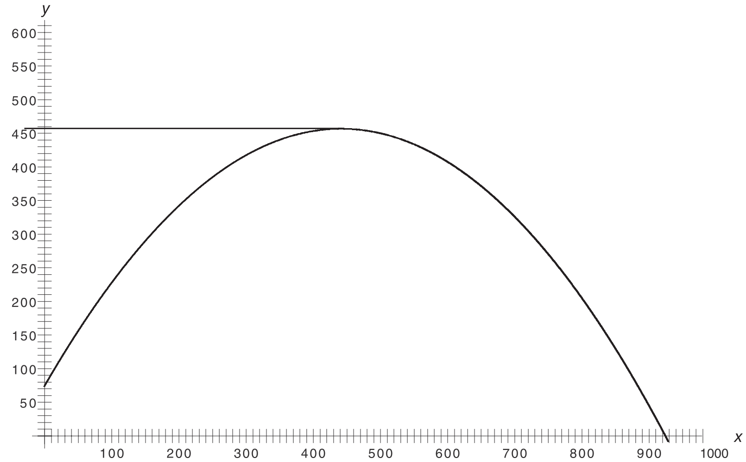
   $y(t) = 75 + 86.6t - (4.9)t^2$.

2. The initial horizontal velocity is 50 meters per second. The horizontal distance is given by $x(t) = 50t$.

3. Solving for *t* gives $t = \frac{x}{50}$. Putting this value of *t* into *y(t)* gives *y* as a function of *x*:

$$y = 75 + \frac{86.6}{50}x - (4.9)\left(\frac{x}{50}\right)^2 \approx 75 + 1.73x - 0.00196x^2$$

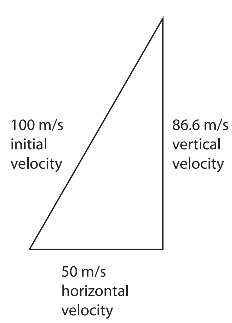
4. Here is a graph showing the path of the ball in the *x*-*y* coordinate system.

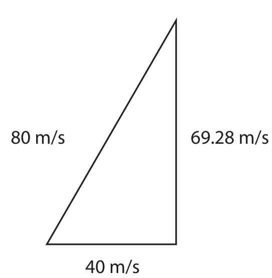


The maximum height the ball reaches (about 460 meters) and where it lands (about 925 meters from the launching position) can be read approximately from the graph.

5. If the ball is thrown at an initial velocity of 80 meters per second at a 60° angle, the triangle would be similar to the original triangle. The ratio of the sides of the first triangle to the sides of the original triangle is 80 to 100 or 0.8 to 1. We can get the needed velocities from the solution to problem 1 by multiplying by 0.8:

86.6 m/s • (0.8) = 69.28 m/s and 50 m/s • (0.8) = 40 m/s





Additional Algebra II TEKS:

(2A.8) Quadratic and square root functions. The student formulates equations and inequalities based on quadratic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

(A) analyze situations involving quadratic functions and formulate quadratic equations or inequalities to solve problems.

(C) compare and translate between algebraic and graphical solutions of quadratic equations.

(D) solve quadratic equations and inequalities using graphs, tables, and algebraic methods.

Notes

Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

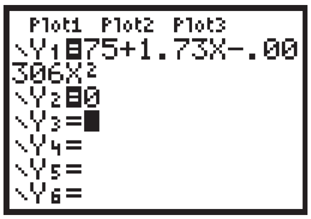
Note: The values for *w* and *v* may also be found by using right triangle trigonometry or special right triangle relationships for a 30-60 right triangle.

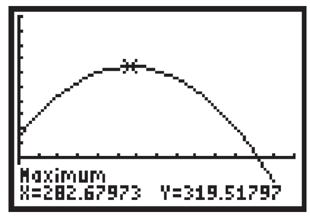
$$y(t) = 75 + 69.28\,t - (4.9)t^2.$$

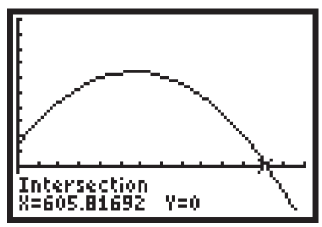
$$x(t) = 40\,t$$

$$y = 75 + \frac{69.28}{40}x - (4.9)\left(\frac{x}{40}\right)^2 \approx 75 + 1.73x - 0.00306x^2$$

6. Here are graphs of the paths:







The maximum height and where the ball hits the ground can be read approximately from the graphs. The maximum height is approximately 320 meters and the ball hits the ground at approximately 606 meters.

Extension Questions:

- Write the equation you found in problem 1 in vertex form to determine the maximum height. How does that compare with the values you found from the graph?

  To determine maximum height algebraically requires finding the vertex of the parabola using completing the square.

$$y = 75 + 1.73x - 0.00196x^2$$
$$y = -0.00196x^2 + 1.73x + 75$$
$$y = -0.00196\left(x^2 - \frac{1.73}{0.00196}x\right) + 75$$
$$y = -0.00196\left(x^2 - \frac{1.73}{0.00196}x + \left(\frac{1}{2}\bullet\frac{1.73}{0.00196}\right)^2\right) + 0.00196\left(\frac{1}{2}\bullet\frac{1.73}{0.00196}\right)^2 + 75$$
$$y = -0.00196\left(x - \frac{1}{2}\bullet\frac{1.73}{0.00196}\right)^2 + 456.75$$
$$y = -0.00196(x - 441.32)^2 + 456.75$$

  The ball reaches a maximum height of approximately 456.75 meters at 441.32 meters from the launch point.

  To determine when the ball hits the ground, let the vertical height be zero.

$$y = 75 + 1.73x - 0.00196x^2$$
$$0 = -0.00196x^2 + 1.73x + 75$$
$$a = -0.00196 \quad b = 1.73 \quad c = 75$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(1.73) \pm \sqrt{(1.73)^2 - 4(-0.00196)75}}{2(-0.00196)}$$
$$x = -41.41 \text{ or } 924.06$$

  The negative solution does not have meaning in this situation. The ball lands approximately 924.06 meters from the tower. These figures are in agreement with the ones obtained from the graphs.

- The triangles described in problems 5 and 6 were similar and the sides were proportional. Are the values determined for maximum height and when the ball hits the ground for the new situation and the original situation proportional? Explain why or why not.

Using the solutions from problems 4 and 6, the ratio for the maximum height is

$$\frac{320}{460} \approx 0.697$$

The ratio of the distances to where the object lands is

$$\frac{606}{925} \approx 0.655$$

The ratios are not the same. Thus, the values are computed using equations that do not represent a proportional relationship.

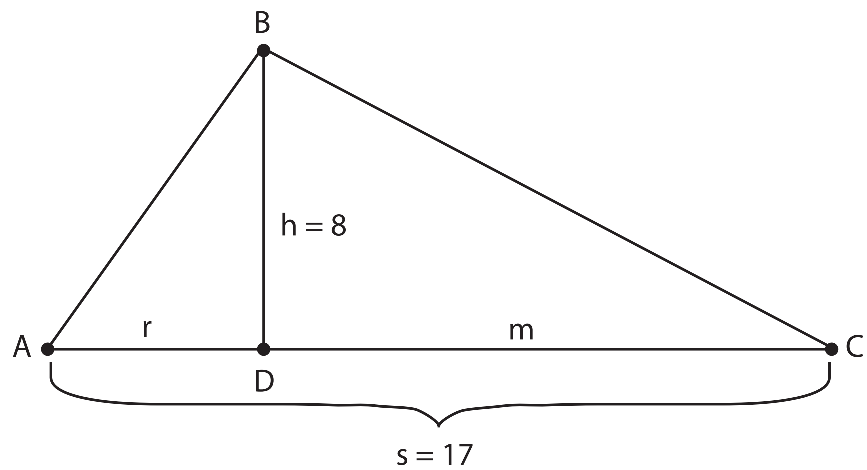
Triangle Solutions

We are given only this information about a right triangle:

- The hypotenuse has length *s* = 17 cm.
- The altitude to the hypotenuse has length *h* = 8 cm.

The altitude divides the hypotenuse into two pieces of lengths that we will call *r* and *m*.

Show how to find r and m from *s* = 17 cm and *h* = 8 cm *using two different methods.*



Notes

Materials:

Straight edge and compass

(or graphing utility)

Graphing calculator

Algebra II TEKS Focus:
(2A.8) Quadratic and square root functions. The student formulates equations and inequalities based on quadratic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

(A) analyze situations involving quadratic functions and formulate quadratic equations or inequalities to solve problems.

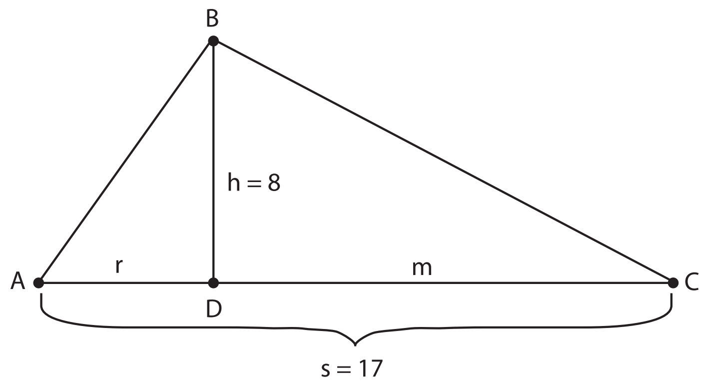
(B) analyze and interpret the solutions of quadratic equations using discriminants and solve quadratic equations using the quadratic formula.

(D) solve quadratic equations and inequalities using graphs, tables, and algebraic methods.

Additional Algebra II TEKS:
None

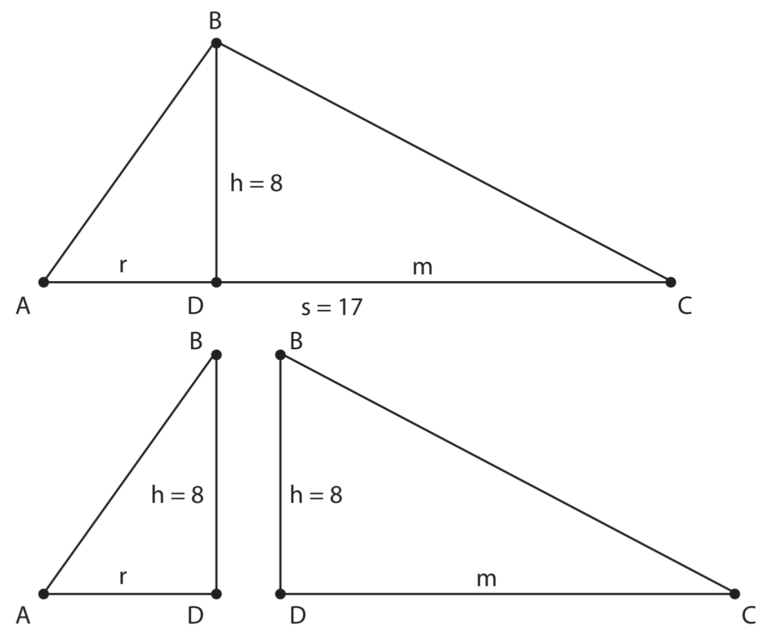
Scaffolding Questions:

- The altitude to the hypotenuse of a right triangle divides the triangle into two smaller triangles. What is true of the relationship among all three triangles?
- What is the length of the altitude in terms of the lengths of the pieces the altitude divides the hypotenuse into?
- In the figure below, find four independent equations relating the variables.



Sample Solutions:

One approach to the problem is to consider the similar triangles.



The altitude to the hypotenuse of a right triangle creates two triangles that are similar to the original triangle.

$$\Delta ADB \sim \Delta BDC \sim \Delta ABC$$

Corresponding sides are proportional.

$$\frac{h}{r} = \frac{m}{h}$$

Therefore, $r \bullet m = (8)^2$.

The sum of the segments of the hypotenuse must equal 17 cm.

$r + m = 17$

$m = 17 - r$

$r \bullet m = 64$

$r(17 - r) = 64$

$17r - r^2 = 64$

$r^2 - 17r + 64 = 0$

Apply the quadratic formula:

$$r = \frac{-(-17) \pm \sqrt{(-17)^2 - 4(1)64}}{2(1)}$$

$r \approx 11.37$ or 5.63

If $r = 11.37$ cm,

$m = 17 - 11.37 = 5.63$ cm.

If $r = 5.63$ cm,

$m = 17 - 5.63 = 11.37$ cm.

Connection to TAKS:

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Objective 9: The student will demonstrate an understanding of percents, proportional relationships, probability, and statistics in application problems.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

Another approach is to use the Pythagorean Theorem. From the initial figure we have a system of four equations. They result from the Pythagorean Theorem and the definition of *r* and *m*, where $p = \overline{AB}$ and $q = \overline{BC}$.

I	II	III	IV
$r^2 + (8)^2 = p^2$	$m^2 + (8)^2 = q^2$	$p^2 + q^2 = (17)^2$	$r + m = 17$

- Eliminate *m* from II using IV.

$$(17 - r)^2 + (8)^2 = q^2$$

- Eliminate *p* from III using I:

$$r + (8)^2 + q^2 = (17)^2$$

- Eliminate q from the preceding two equations:

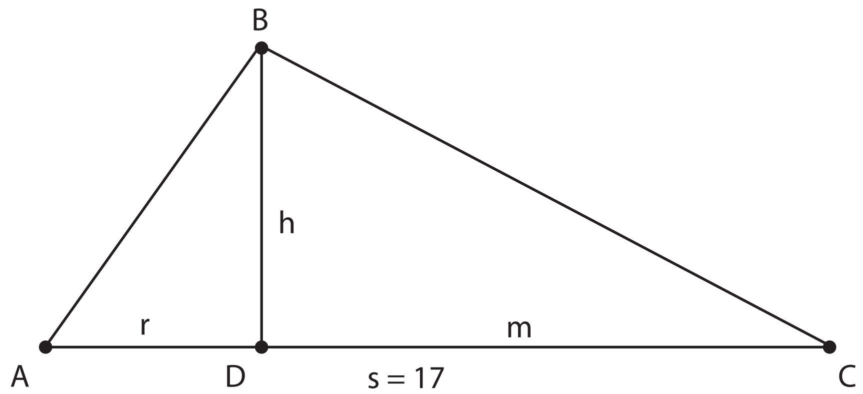
$$(17 - r)^2 + (8)^2 = (17)^2 - r^2 - (8)^2$$

- Simplify $r^2 - 17\,r + (8)^2 = 0$.

This is the same equation as we found in the first method. It has roots 5.63 cm. or 11.37 cm.

Extension Questions:

- Assume that the hypotenuse of the right triangle is 17, but the height is not known. What is the range of values for the altitude to the hypotenuse, *h*?



$$\frac{r}{h} = \frac{h}{m}$$
$$h^2 = r\,m$$

$$r + m = 17$$
$$m = 17 - r$$
$$r\,(17 - r) = h^2$$

$$r^2 - 17r + h^2 = 0$$
$$r = \frac{17 \pm \sqrt{17^2 - 4(1)h^2}}{2(1)}$$

This equation has a solution only if the radicand is not negative.

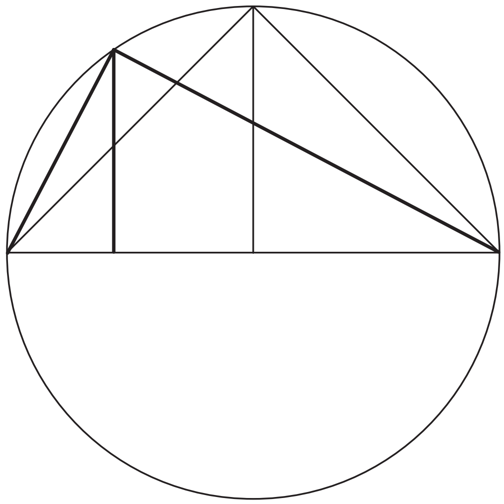
$$17^2 - 4(1)h^2 \geq 0$$
$$17^2 \geq 4(1)h^2$$
$$h^2 \leq \frac{17^2}{4}$$
$$-\frac{17}{2} \leq h \leq \frac{17}{2}$$

Since h is the length of the hypotenuse, it cannot be negative.

$$0 < h \leq \frac{17}{2}$$

Another approach is to consider the problem geometrically. A right triangle can be inscribed in a semicircle. The diameter of the circle is the length of the hypotenuse of the circle.

The maximum height would be the radius of the circle, which is one-half of the hypotenuse, $\frac{17}{2}$ *units:* $0 < h \leq \frac{17}{2}$.



- What is the maximum area that the triangle could have?

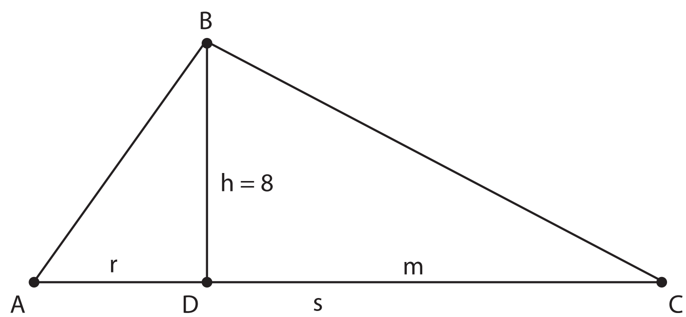
  The formula for the area of a triangle is

$$\begin{aligned} area &= \frac{1}{2} \bullet \text{base} \bullet \text{height} \\ &= \frac{1}{2} \bullet 17 \bullet h \end{aligned}$$

  The maximum area will occur when the height is at its maximum value, $\frac{h}{2}$ *.*

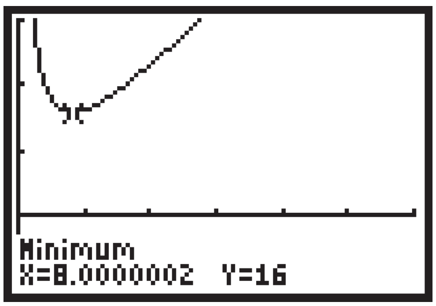
$$\begin{aligned} area &= \frac{1}{2} \bullet 17 \bullet \frac{17}{2} \\ &= \frac{289}{4}\ \text{cm}^2 \end{aligned}$$

- Suppose the height of the right triangle *h* is fixed at 8 cm. What is the range of possible values for the hypotenuse of the triangle?



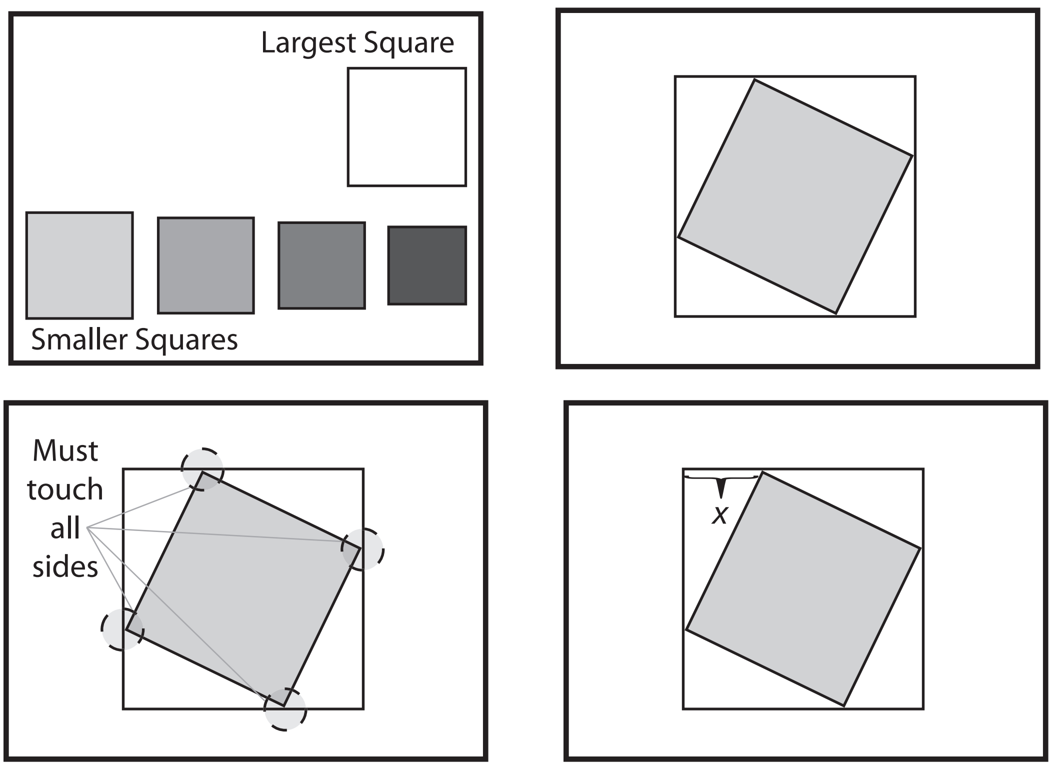
$$\frac{r}{8} = \frac{8}{m}$$
$$r\,m = 64$$
$$r = \frac{64}{m}$$
$$s = r + m$$
$$s = \frac{64}{m} + m$$

Graph the function $y = \frac{64}{x} + x$, where y represents the hypotenuse length and x represents the length of a segment of the hypotenuse. Find the minimum value. The minimum value occurs at x = 8 and y = 16. The least value of the hypotenuse is 16 centimeters. There is no upper bound on the hypotenuse.



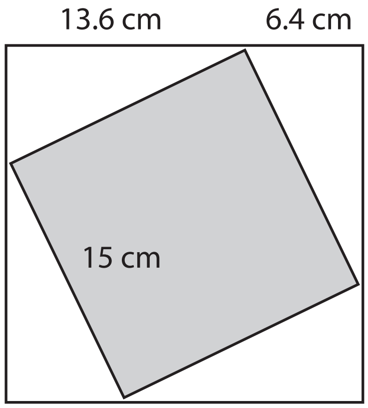
Spinning Square

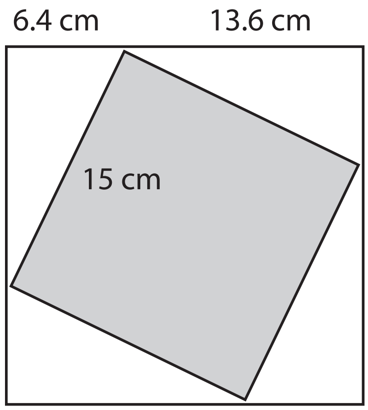
Karen discovers a package of different-sized square sheets of paper. As her sister, Mariana, watches, Karen makes designs with the sheets by placing the smaller squares, one at a time, on top of the largest square, as shown below. As Karen plays, she makes sure that each corner of the smaller square touches a different side of the larger square.



The largest square is 20 cm per side. The *x* shown in the diagram above is measured from the corner of the largest square to a corresponding corner of the smaller square. Mariana realizes that the value of *x* will be different for each size of square that is placed on the largest square. To determine the relationship, she decides to measure the length of the side of the smaller square and the length of *x*. The results of her data collection are shown in the table following the diagrams on the next page.

She finds that for each square there are actually two values of *x* that can be collected. These are shown on two different lines of the table.





Determine the area of the inner square and the ratio of the areas of the two squares for each of the sets of data collected. Record your results in the table on the following page.

Length of the side of the inner square in cm	***x* in cm**	**Area of the inner square in cm**	**Ratio of the area of the inner square to the area of the outer square**
15	6.4		
15	13.6		
15.5	5.5		
15.5	14.5		
16	4.7		
16	15.3		
17	3.3		
17	16.7		
18	2.1		
18	17.9		
19	1.1		
19	18.9		
20	0		

1. Create a scatterplot of the set of ordered pairs (the measure of *x*, the ratio of the area of the inner square to the area of the largest square).

2. Find a model for the relationship between the length of *x* and the ratio of the area of the smaller square to the large square. Graph your model with the data to judge its reasonableness.

3. Describe the domain and range for the problem situation. How are these the same or different for the domain and range of the model?

4. Use your model to determine the ratio of small square to large square when $x = 3.8$. Explain the meaning of your answer in the context of the problem.

5. For what value(s) of *x* will the small square cover at least 75% of the large square?

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus:
(2A.1) Foundations for functions. The student uses properties and attributes of functions and applies functions to problem situations.

The student is expected to:

(A) identify the mathematical domains and ranges of functions and determine reasonable domain and range values for continuous and discrete situations.

(B) collect and organize data, make and interpret scatterplots, fit the graph of a function to the data, interpret the results, and proceed to model, predict, and make decisions and critical judgments.

Scaffolding Questions:

- Can one of the small squares be placed in the outer square in more than one way?
- What is the relationship between the two possible *x*-values for any square?
- If one of the inner squares were the exact size as the outer square, when it was placed inside the outer square what values would be given to *x*?
- When the inner square is placed in the outer square, four triangles are formed. How are they related to each other?
- What are the lengths of the sides of the four triangles that are formed? Express your answer in terms of *x*.

Sample Solutions:

The completed table is shown opposite:

Length of the side of the inner square in cm	*x* in cm	Area of the inner square in cm	Ratio of the area of the inner square to the area of the outer square
15	6.4	225	0.5625
15	13.6	225	0.5625
15.5	5.5	240.45	0.60063
15.5	14.5	240.45	0.60063
16	4.7	256	0.64
16	15.3	256	0.64
17	3.3	289	0.7225
17	16.7	289	0.7225
18	2.1	324	0.81
18	17.9	324	0.81
19	1.1	361	0.9025
19	18.9	361	0.9025
20	0	400	1

(2A.6) Quadratic and square root functions. The student understands that quadratic functions can be represented in different ways and translates among their various representations.

The student is expected to:

(A) determine the reasonable domain and range values of quadratic functions, as well as interpret and determine the reasonableness of solutions to quadratic equations and inequalities.

(B) relate representations of quadratic functions, such as algebraic, tabular, graphical, and verbal descriptions.

Additional Algebra II TEKS:
(2A.3) Foundations for functions. The student formulates systems of equations and inequalities from problem situations, uses a variety of methods to solve them, and analyzes the solutions in terms of the situations.

The student is expected to:

(B) use algebraic methods, graphs, tables, or matrices, to solve systems of equations or inequalities.

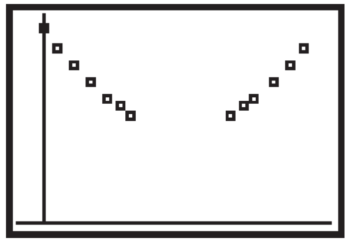
Notes

(2A.8) Quadratic and square root functions. The student formulates equations and inequalities based on quadratic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

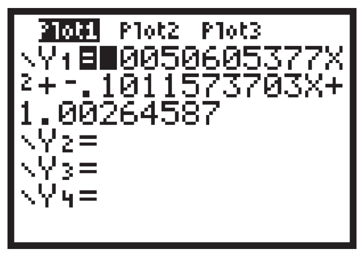
(A) analyze situations involving quadratic functions and formulate quadratic equations or inequalities to solve problems.

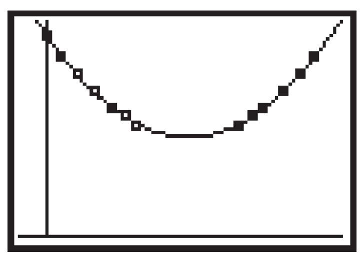
1. The data was entered into the lists of a graphing calculator and a scatterplot was created.



2. Several approaches can be taken to determine a model.

The regression tool on a graphing calculator resulted in the following rule and graph:





Another method to find a quadratic model is to use three points and the equation $y = ax^2 + bx + c$.

Use the points (0, 1), (2.1, 0.81), (4.7, 0.64).

$$y = ax^2 + bx + c$$
$$x = 0,\quad y = 1$$
$$1 = c$$

The rule becomes $y = ax^2 + bx + 1$.

Substitute the other points in the rule and solve for *a* and *b*.

$$0.81 = a(2.1)^2 + b(2.1) + 1 \qquad 0.64 = a(4.7)^2 + b(4.7) + 1$$
$$4.41a + 2.1b = -0.19 \qquad 22.09a + 4.7b = -0.36$$

This system of equations was solved using a matrix.

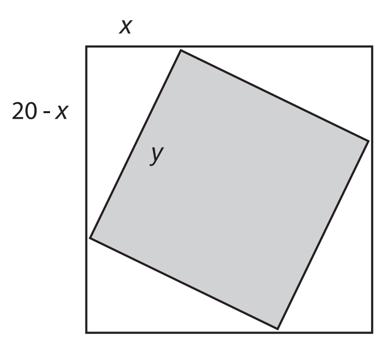
$$A = \begin{pmatrix} 4.41 & 2.1 \\ 22.09 & 4.7 \end{pmatrix} \qquad B = \begin{pmatrix} -0.19 \\ -0.36 \end{pmatrix} \qquad X = \begin{pmatrix} a \\ b \end{pmatrix}$$
$$AX = B$$
$$X = A^{-1}B$$
$$X = \begin{pmatrix} 0.005 \\ -0.102 \end{pmatrix}$$

The rule is $y = 0.005x^2 - 0.102x + 1$.

Another way to construct the model is to analyze the geometric properties of the resulting figure.

For a given value of *x*, find a formula for the area of the inner square.

The three sides of any of the four triangles formed are *x*, 20 – *x*, and the hypotenuse of the square, *y*.



The Pythagorean Theorem may be used to find the relationship between the three sides of the triangle and to solve for *y*.

Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

$$y^2 = x^2 + (20 - x)^2$$
$$y = \sqrt{x^2 + (20 - x)^2}$$
$$y = \sqrt{2x^2 - 40x + 400}$$

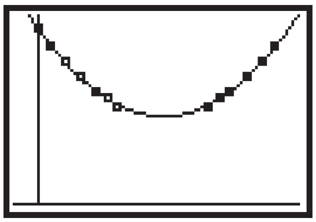
The hypotenuse of the triangle, y, is equal to one side of the inner square. Thus, the area of the inner square with respect to x is y^2 or $2x^2 - 40x + 400$ square centimeters.

The area of the large square is 20 squared or 400 square centimeters. The ratio of the area of the inner square to that of the outer square with respect to x is

$$\frac{\text{area of inner square}}{\text{area of outer square}} = \frac{2x^2 - 40x + 400}{400}$$
$$= 0.005x^2 - 0.1x + 1$$

The function is $y = 0.005x^2 - 0.1x + 1$.

The graph of the model fits the scatterplot.



3. Domain $0 \leq x \leq 20$

   The value of x could be 0 or 20 if the inner square is the same size as the large square.

   Range $0.5 \leq y \leq 1$

   From the data it seems that the value of the ratio must be more than 0.5. The ratio must be less than or equal to one.

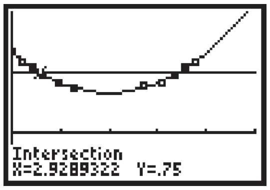
4. Use the rule $y = 0.005x^2 - 0.1x + 1$ to determine the value of y when x = 3.8.

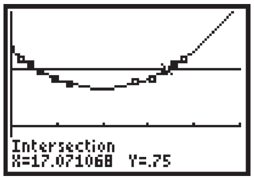
$$y = 0.005x^2 - 0.1x + 1$$
$$y = 0.005(3.8)^2 - 0.1(3.8) + 1$$
$$y = 0.6922$$

   The ratio of the two areas is 0.6922. This means that a square that is placed on top of the square with an x measure of 3.8 centimeters will cover about 69% of the larger square.

5. To ask when it covers at least 75% is to ask when the ratio is greater than or equal to 0.75.

   Graph the model $y = 0.005x^2 - 0.1x + 1$ and the line $y = 0.75$.



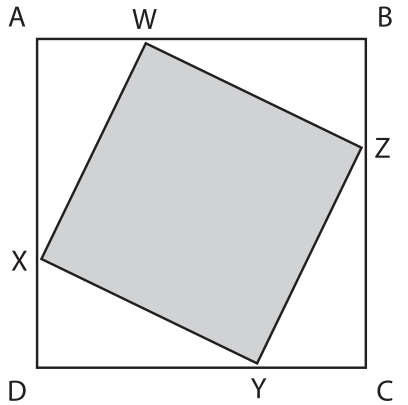


   The ratio of the areas will be greater than or equal to 0.75 for $0 \leq x \leq 2.9289$ or for $17.071 \leq x \leq 20$.

Extension Questions:

Note to teacher: Each of the methods shown in the solution should be a part of your class discussions about this problem. Ask the students to develop the formula by any method that was not addressed in your class summary of this problem.

- Explain why the triangles shown in the diagram are congruent.



The four triangles are right triangles: $\angle A \cong \angle B \cong \angle C \cong \angle D$

AW = BZ = YC = DX = x

WX = XY = YZ = ZW = y

The four right triangles are congruent because the hypotenuses are congruent and the shorter legs are congruent.

Torricelli's Law

There are two water tanks, side by side. Tank 1 has 225 liters of water, while Tank 2 has 900 liters of water.

At noon, a pump starts to drain Tank 1 and finishes 5 hours later. The rate of pumping in liters per hour is constant.

Also at noon, Tank 2 starts to drain. But for this tank, no pump is used. Instead, the water drains by itself through an opening made in the bottom. This tank also drains completely in exactly 5 hours.

In order to solve this problem, you need to know that a tank draining by itself (the way Tank 2 does) does <u>not</u> drain at a constant rate (the way Tank 1 does), but instead follows Torricelli's Law:

> If it takes a time T to fully drain a tank of volume V through an opening in the bottom, then the volume of water in the tank as a function of time t is
>
> $$V_0 \bullet \left(1 - \frac{t}{T}\right)^2$$

Find to the nearest minute all times at which the two tanks hold exactly the same amount of water.

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus: (2A.8) Quadratic and square root functions. The student formulates equations and inequalities based on quadratic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

(A) analyze situations involving quadratic functions and formulate quadratic equations or inequalities to solve problems.

(B) analyze and interpret the solutions of quadratic equations using discriminants and solve quadratic equations using the quadratic formula.

(C) compare and translate between algebraic and graphical solutions of quadratic equations.

(D) solve quadratic equations and inequalities using graphs, tables, and algebraic methods.

Additional Algebra II TEKS: (2A.6) Quadratic and square root functions. The student understands that quadratic functions can be represented in different ways

Scaffolding Questions:

- If the volume of water in a tank decreases at a constant rate from some initial value to zero, what does the graph of volume as a function of time look like?
- If the volume of water in a tank decreases at a constant rate from 225 liters to zero in 5 hours, what is this constant rate?
- If the volume of water in a tank decreases at a constant rate from 225 liters to zero in 5 hours, what is a formula for volume as a function of time?
- For a tank draining according to Torricelli's Law from 900 liters to zero in 5 hours, graph the volume as a function of time.
- How could you use your graphs of two tanks draining to see when they have the same volume?
- How could you use the formulas for two tanks draining to see when they have the same volume?

Sample Solutions:

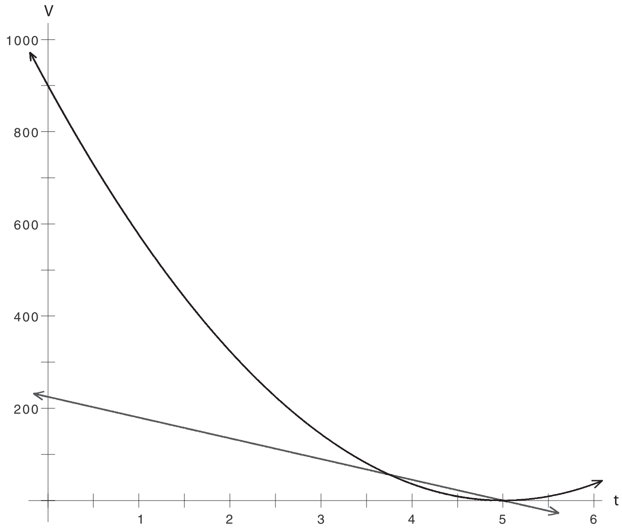
For Tank 1, we know that 225 liters is pumped out at a constant rate in 5 hours, so this rate must be 45 liters per hour. A function giving the volume remaining in the tank as a function of time is

$$f(t) = 225 - 45t$$

For Tank 2, we know that 900 liters drains in 5 hours according to Torricelli's Law. A function giving the volume remaining in the tank as a function of time is

$$g(t) = 900 \bullet \left(1 - \frac{t}{5}\right)^2$$

Graphing these two functions together, we have



The graph shows that the two volumes are the same at <u>approximately</u> time t = 3.7 hours and also at time t = 5 hours. But to get the first time to the nearest minute we may get a more accurate answer by using symbolic methods

One way is to solve for t the equation that sets the values of f and g equal:

$$225 - 45t = 900 \bullet \left(1 - \frac{t}{5}\right)^2$$

This leads to a quadratic equation in t:

$$4t^2 - 35t + 75 = 0$$

The solutions are t = 3.75 and t = 5, agreeing with what we saw on the graph.

3.75 hours is 3.75 times 60 minutes or 1,350 minutes.

5 hours is 300 minutes.

and translates among their various representations.

The student is expected to:

(A) determine the reasonable domain and range values of quadratic functions, as well as interpret and determine the reasonableness of solutions to quadratic equations and inequalities.

Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

Extension Questions:

- Solve this problem in general: If a tank with volume V drains in T hours according to Torricelli's Law, and another tank with volume rV (where $r < 1$) is pumped dry at a uniform rate in T hours (starting at the same time), when will the two volumes be the same?

  Sample Solution:

  Solve for t:

$$V(1 - \frac{t}{T})^2 = rV\left(1 - \frac{t}{T}\right)$$

$$\left(1 - \frac{t}{T}\right)^2 = r\left(1 - \frac{t}{T}\right)$$

$$\left(1 - \frac{t}{T}\right)^2 - r\left(1 - \frac{t}{T}\right) =$$

$$\left(1 - \frac{t}{T}\right)\left[\left(1 - \frac{t}{T}\right) - r\right] = 0$$

$$1 - \frac{t}{T} = 0 \quad \text{or} \quad \left(1 - \frac{t}{T}\right) - r$$

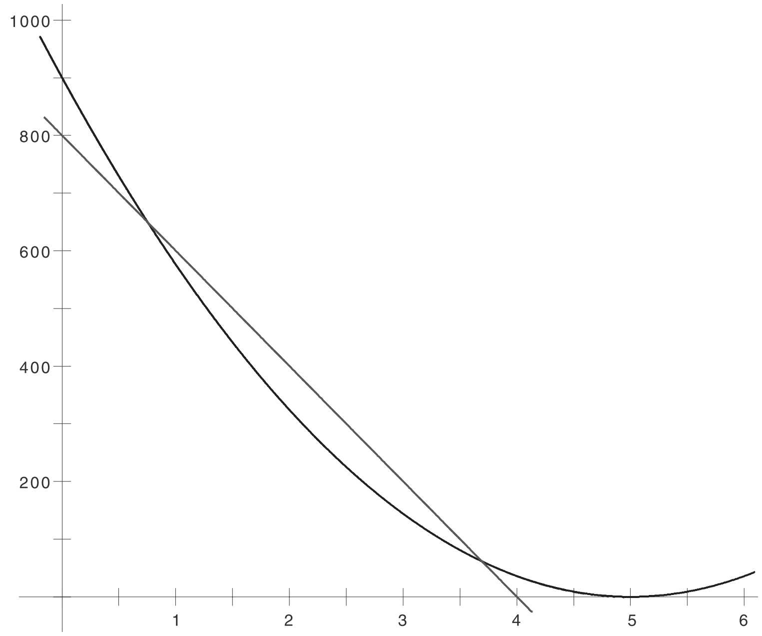
$$T - t = 0 \text{ or } T - t - rT = 0$$

$$t = T \quad \text{or} \quad t = T - rT$$

- Consider Tank 2, draining 900 liters in 5 hours according to Torricelli's Law. Tank 3 starts at the same time with volume V_3 and is pumped dry (at a constant rate) in time T_3. Find values of V_3 and T_3 such that there are two different (non-zero) times when the two tanks have the same (non-zero) volume. Are these values unique?

Sample solution:

Here is a graph of the volume over time of Tank 3. This tank pumps out 800 liters at a constant rate in 4 hours. The two intersections of the two graphs show that there are two places where the tanks have the same volume. By shifting the slope and/or the intercepts of the line, we could find many other cases.



Doing What Mathematicians Do

Mathematicians frequently explore the conclusions that can be drawn regarding solutions of certain equations when one of the equation's constants is allowed to vary.

1. Consider quadratic equations of the form $x^2 + c = 0$, where c is an integer.

   a. Choose a value of c and give the corresponding solution(s) so that the equation $x^2 + c = 0$ has

      i. two real solutions.

      ii. exactly one real solution.

      iii. no real solutions.

   b. What must be true about c in order for the equation $x^2 + c = 0$ to have

      i. two real solutions?

      ii. exactly one real solution?

      iii. no real solutions?

2. Consider quadratic equations of the form $x^2 + 2x + c = 0$, where c is an integer.

   a. Choose a value of c and give the corresponding solution(s) so that the equation $x^2 + 2x + c = 0$ has

      i. two real solutions.

      ii. exactly one real solution.

      iii. no real solutions.

   b. What must be true about c in order for the equation $x^2 + 2x + c = 0$ to have

      i. two real solutions?

      ii. exactly one real solution?

      iii. no real solutions?

3. Consider quadratic equations of the form $x^2 + bx + 9 = 0$, where b is an integer.

   a. Choose a value of b and give the corresponding solution(s) so that the equation $x^2 + bx + 9 = 0$ has

      i. two real solutions.

      ii. exactly one real solution.

      iii. no real solutions.

   b. What must be true about b in order for the equation $x^2 + bx + 9 = 0$ to have

      i. two real solutions?

      ii. exactly one real solution?

      iii. no real solutions?

4. Consider quadratic equations of the form $ax^2 + 2x + 1 = 0$, where a is an integer.

   a. Choose a value of a and give the corresponding solution(s) so that the equation $ax^2 + 2x + 1 = 0$ has

      i. two real solutions.

      ii. exactly one real solution.

      iii. no real solutions.

   b. What must be true about a in order for the equation $ax^2 + 2x + 1 = 0$ to have

      i. two real solutions?

      ii. exactly one real solution?

      iii. no real solutions?

5. For any quadratic equation of the form $ax^2 + bx + c = 0$, the expression $b^2 - 4ac$ is called the "discriminant."

a. Where do you find the discriminant in the quadratic formula?

b. What must be true about the discriminant in order for the quadratic equation $ax^2 + bx + c = 0$ to have

i. two real solutions?

ii. exactly one real solution?

iii. no real solutions?

Notes

Materials:

None required.

Algebra II TEKS Focus:
(2A.2) Foundations for functions. The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

The student is expected to:

(B) use complex numbers to describe the solutions of quadratic equations.

(2A.8) Quadratic and square root functions. The student formulates equations and inequalities based on quadratic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

(B) analyze and interpret solutions of quadratic equations using discriminants and solve quadratic equations using the quadratic formula.

Additional Algebra II TEKS:
None

Scaffolding Questions:

- What does *c* (or *a* or *b*) represent in the equation?
- What are some examples of numbers that *c* (or *a* or *b*) could represent?
- How can you solve quadratic equations of the form $x^2 + c = 0$?
- How can you solve quadratic equations of the form $ax^2 + bx + c = 0$?
- What are some examples of numbers that are not real?
- For what values of *c* will $\sqrt{c}$ be a real number? For what values of *c* will $\sqrt{c}$ not be a real number?
- What part of the quadratic formula will determine whether the solutions are real?

Sample Solutions:

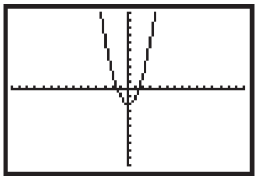
1.a. i. For example: c = -2, -4, or -10.

Corresponding solutions:

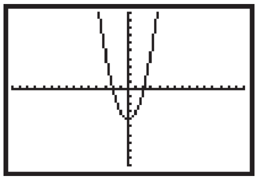
$x^2 - 2 = 0$	$x^2 - 4 = 0$	$x^2 - 10 = 0$
$x^2 = 2$	$x^2 = 4$	$x^2 = 10$
$x = \pm\sqrt{2}$	$x = \pm\sqrt{4} = \pm 2$	$x = \pm\sqrt{10}$

The graph shows that in each case the related function intersects the *x*-axis in two points.

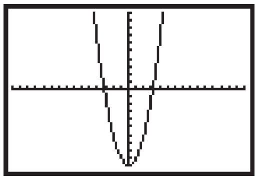
$f(x) = x^2 - 2$



$f(x) = x^2 - 4$



$f(x) = x^2 - 10$

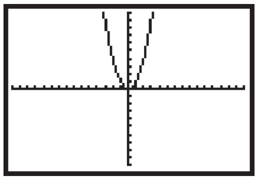


ii. Only when $c = 0$.

Corresponding solution:

$$x^2 + 0 = 0$$
$$x^2 = 0$$
$$x = 0$$

The graph of the related function intersects the x-axis in one point (0, 0).



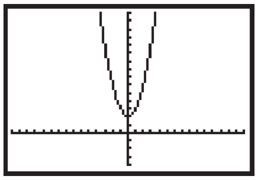
iii. For example: $c = 2$, 4, or 10.

Corresponding solutions:

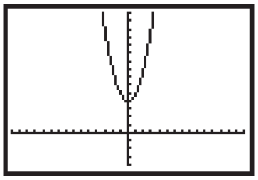
$x^2 + 2 = 0$	$x^2 + 4 = 0$	$x^2 + 10 = 0$
$x^2 = -2$	$x^2 = -4$	$x^2 = -10$
$x = \pm\sqrt{-2} = \pm i\sqrt{2}$	$x = \pm\sqrt{-4} = \pm 2i$	$x = \pm\sqrt{-10} = \pm i\sqrt{10}$

The graph shows that in each case the related function does not intersect the x-axis.

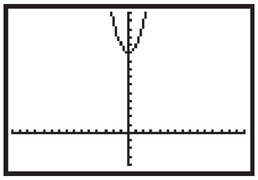
$f(x) = x^2 + 2$



$f(x) = x^2 + 4$



$f(x) = x^2 + 10$



b) $x^2 + c = 0$

$x^2 = -c$

$x = \pm\sqrt{-c}$

i. Two real solutions when $c < 0$.
ii. Exactly one real solution when $c = 0$.
iii. No real solutions when $c > 0$.

Connection to TAKS:

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

2. a. i. For example: c = 0, -2, or -3.

Corresponding solutions:

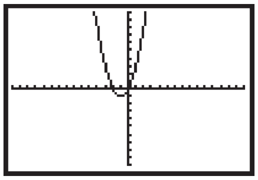
$$x^2+2x+0=0 \qquad x(x+2)=0 \qquad x=0 \text{ or } -2$$

$$x^2+2x-2=0$$
$$x=\frac{-2\pm\sqrt{2^2-4(1)(-2)}}{2(1)}=\frac{-2\pm2\sqrt{3}}{2}=-1\pm\sqrt{3}$$

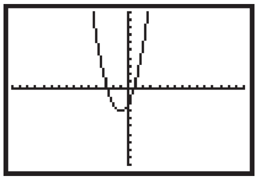
$$x^2+2x-3=0 \qquad (x+3)(x-1)=0 \qquad x=-3 \text{ or } 1$$

Graphs of the functions show the two solutions in each of these situations.

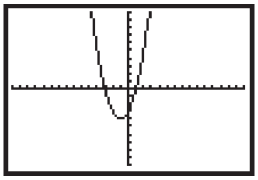
$f(x)=x^2+2x+0$



$f(x)=x^2+2x-2$



$f(x)=x^2+2x-3$



ii. Only when c = 1.

Corresponding solution:

$$x^2+2x+1=0$$
$$(x+1)^2=0$$
$$x=-1$$

iii. For example: c = 2, 4, or 10.

Corresponding solutions:

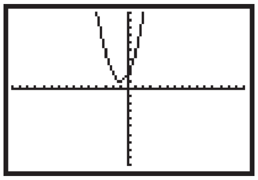
$$x^2+2x+2=0$$
$$x=\frac{-2\pm\sqrt{2^2-4(1)(2)}}{2(1)}=\frac{-2\pm\sqrt{-4}}{2}=\frac{-2\pm2i}{2}=-1\pm i$$

$$x^2+2x+4=0$$
$$x=\frac{-2\pm\sqrt{2^2-4(1)(4)}}{2(1)}=\frac{-2\pm\sqrt{-12}}{2}=\frac{-2\pm2i\sqrt{3}}{2}=-1\pm i\sqrt{3}$$

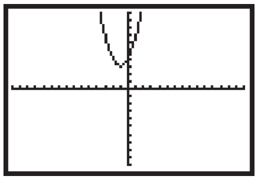
$$x^2+2x+10=0$$
$$x=\frac{-2\pm\sqrt{2^2-4(1)(10)}}{2(1)}=\frac{-2\pm\sqrt{-36}}{2}=\frac{-2\pm6i}{2}=-1\pm3i$$

Graphs of the functions show the two solutions in each of these situations.

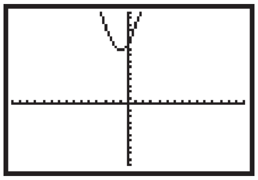
$f(x) = x^2 + 2x + 2$



$f(x) = x^2 + 2x + 4$



$f(x) = x^2 + 2x + 10$



b) $x^2 + 2x + c = 0$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(c)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4 - 4c}}{2}$$

i. Two real solutions when $4 - 4c > 0$ or $c < 1$.

ii. Exactly one real solution when $4 - 4c = 0$ or $c = 1$.

iii. No real solutions when $4 - 4c < 0$ or $c > 1$.

3. a. i. For example: b = -7, 7, or 10.

Corresponding solutions:

$x^2 - 7x + 9 = 0$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(9)}}{2(1)}$$

$$x = \frac{7 \pm \sqrt{13}}{2}$$

$x^2 + 7x + 9 = 0$

$$x = \frac{-7 \pm \sqrt{7^2 - 4(1)(9)}}{2(1)}$$

$$x = \frac{-7 \pm \sqrt{13}}{2}$$

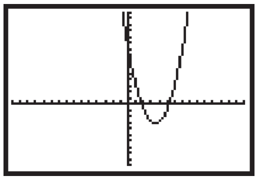
$x^2 + 10x + 9 = 0$

$(x+1)(x+9) = 0$

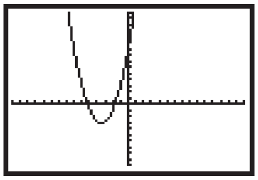
$x = -1$ or -9

Graphs of the functions show the two solutions in each of these situations.

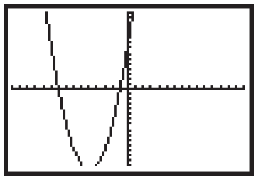
$f(x) = x^2 - 7x + 9$



$f(x) = x^2 + 7x + 9$



$f(x) = x^2 + 10x + 9$



ii. Only when b = -6 or 6.

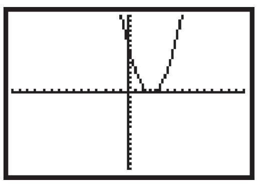
Corresponding solutions:

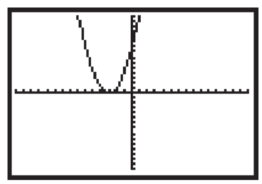
$$x^2 - 6x + 9 = 0 \qquad\qquad x^2 + 6x + 9 = 0$$
$$(x-3)^2 = 0 \qquad\qquad (x+3)^2 = 0$$
$$x = 3 \qquad\qquad x = -3$$

The graphs of the two functions intersect the x-axis in one point.

$f(x) = x^2 - 6x + 9$ $\qquad$ $f(x) = x^2 + 6x + 9$

(3, 0) $\qquad$ (-3, 0)





iii. For example: b = -2, 0, or 3.

Corresponding solutions:

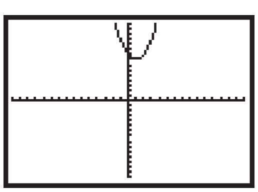
$$x^2 - 2x + 9 = 0$$
$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(9)}}{2(1)}$$
$$= \frac{2 \pm 4i\sqrt{2}}{2}$$
$$= 1 \pm 2i\sqrt{2}$$

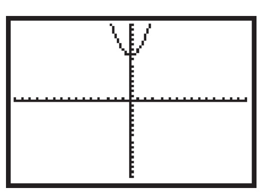
$$x^2 + 9 = 0$$
$$x^2 = -9$$
$$x = \pm\sqrt{-9}$$
$$x = \pm 3i$$

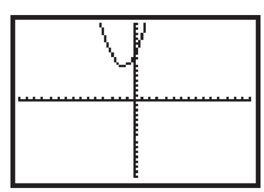
$$x^2 + 3x + 9 = 0$$
$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(9)}}{2(1)}$$
$$= \frac{-3 \pm 3i\sqrt{3}}{2}$$
$$= -\frac{3}{2} \pm \frac{3}{2} i\sqrt{3}$$

The graphs of these functions do not intersect the x-axis.

$f(x) = x^2 - 2x + 9$ $\qquad$ $f(x) = x^2 + 9$ $\qquad$ $f(x) = x^2 + 3x + 9$







b) $x^2 + bx + 9 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4(1)(9)}}{2(1)}$$

$$= \frac{-b \pm \sqrt{b^2 - 36}}{2}$$

i. Two real solutions when $b^2 - 36 > 0$ $\quad b < -6$ or $b > 6$.

ii. Exactly one real solution when $b^2 - 36 = 0$ $\quad b = 6$ or -6.

iii. No real solutions when $b^2 - 36 < 0$ $\quad -6 < b < 6$.

4. a. i. For example: a = -1, -4, or -5.

Corresponding solutions:

$$-x^2 + 2x + 1 = 0$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(-1)(1)}}{2(-1)}$$

$$= \frac{-2 \pm 2\sqrt{2}}{-2}$$

$$= 1 \pm \sqrt{2}$$

$$-4x^2 + 2x + 1 = 0$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(-4)(1)}}{2(-4)}$$

$$= \frac{-2 \pm 2\sqrt{5}}{-8}$$

$$= \frac{1}{4} \pm \frac{1}{4}\sqrt{5}$$

$$-5x^2 + 2x + 1 = 0$$

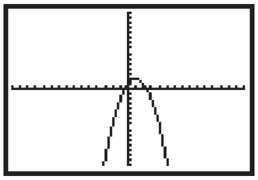
$$x = \frac{-2 \pm \sqrt{2^2 - 4(-5)(1)}}{2(-5)}$$

$$= \frac{-2 \pm 2\sqrt{6}}{-10}$$

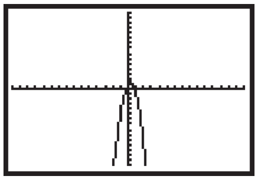
$$= \frac{1}{5} \pm \frac{1}{5}\sqrt{6}$$

The graphs of these functions intersect the x-axis in two points.

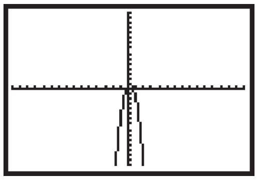
$f(x) = -x^2 + 2x + 1$



$f(x) = -4x^2 + 2x + 1$



$f(x) = -5x^2 + 2x + 1$



ii. Only when a = 1. (Note: If a = 0, the equation is no longer quadratic.)

Corresponding solution:

$$x^2 + 2x + 1 = 0$$

$$(x + 1)^2 = 0$$

$$x = -1$$

iii. For example: a = 2, 4, or 5.

Corresponding solutions:

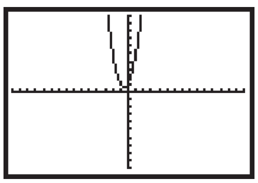
$$2x^2 + 2x + 1 = 0$$
$$x = \frac{-2 \pm \sqrt{2^2 - 4(2)(1)}}{2(2)}$$
$$= \frac{-2 \pm 2i}{4}$$
$$= -\frac{1}{2} \pm \frac{1}{2}i$$

$$4x^2 + 2x + 1 = 0$$
$$x = \frac{-2 \pm \sqrt{2^2 - 4(4)(1)}}{2(4)}$$
$$= \frac{-2 \pm 2i\sqrt{3}}{8}$$
$$= -\frac{1}{4} \pm \frac{1}{4}i\sqrt{3}$$

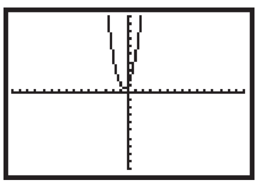
$$5x^2 + 2x + 1 = 0$$
$$x = \frac{-2 \pm \sqrt{2^2 - 4(5)(1)}}{2(5)}$$
$$= \frac{-2 \pm 4i}{10}$$
$$= -\frac{1}{5} \pm \frac{2}{5}i$$

In these situations the graph of the related function does not intersect the x-axis.

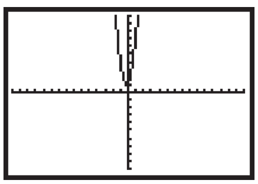
$f(x) = 2x^2 + 2x + 1$



$f(x) = 4x^2 + 2x + 1$



$f(x) = 5x^2 + 2x + 1$



b) $ax^2 + 2x + 1 = 0$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(a)(1)}}{2(a)}$$
$$= \frac{-2 \pm \sqrt{4 - 4a}}{2a}$$

i. Two real solutions when $4 - 4a > 0$ or $a < 1$ (except $a = 0$).

ii. Exactly one real solution when $4 - 4a = 0$ or $a = 1$.

iii. No real solutions when $4 - 4a < 0$ or $a > 1$.

5\. a. The discriminant, $b^2 - 4ac$, is the expression under the radical:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

b. i. Two real solutions when $b^2 - 4ac > 0$.

ii. Exactly one real solution when $b^2 - 4ac = 0$.

iii. No real solutions when $b^2 - 4ac < 0$.

Extension Questions:

- In each problem, you were asked to determine when certain quadratic equations have two real solutions, exactly one real solution, or no real solutions. Is there a graphical way to determine when quadratic equations have two real solutions? Exactly one real solution? No real solutions?

  Solutions for quadratic equations of the form $ax^2 + bx + c = 0$ *will correspond to the x-intercepts of the graph of* $y = ax^2 + bx + c$*. Therefore, when the graph of* $y = ax^2 + bx + c$ *has two x-intercepts, the equation* $ax^2 + bx + c = 0$ *will have two real solutions. When the graph of* $y = ax^2 + bx + c$ *has exactly one x-intercept, the equation* $ax^2 + bx + c = 0$ *will have exactly one real solution. When the graph of* $y = ax^2 + bx + c$ *has no x-intercepts, the equation* $ax^2 + bx + c = 0$ *will have no real solutions.*

Fixed Area Rectangles

You want to fence in a rectangular plot with an area of exactly 360 square meters.

1. Using diagrams, functions, and graphs, determine the possible dimensions if you can use at most 100 meters of fence.

2. Determine the dimensions of the plot with the shortest perimeter.

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus: (2A.8) Quadratic and square root functions. The student formulates equations and inequalities based on quadratic functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student is expected to:

(A) analyze situations involving quadratic functions and formulate quadratic equations or inequalities to solve problems.

(B) analyze and interpret the solutions of quadratic equations using discriminants and solve quadratic equations using the quadratic formula.

(C) compare and translate between algebraic and graphical solutions of quadratic equations.

(D) solve quadratic equations and inequalities using graphs, tables, and algebraic methods.

Additional Algebra II TEKS:
None

Scaffolding Questions:

- Draw a diagram of some possible rectangular plots that have an area of 360 square meters.
- If the length of one side of the plot is 40 meters and the area is 360 square meters, draw the plot.

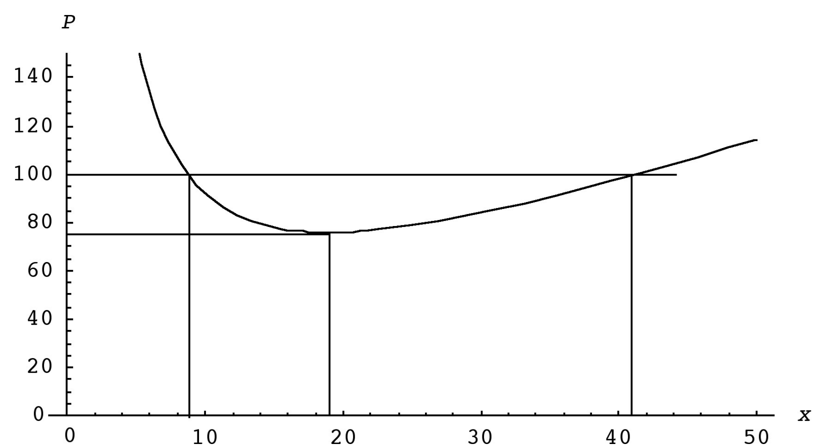
  What is the length of the other side? What is the perimeter of the plot?
- If the length of one side of the plot is *x* meters and the area is 360 square meters, what is the length of the other side? What is the perimeter?
- Graph the perimeter of the plot in terms of *x*.

Sample Solutions:

1. Since the desired area of the rectangle is 360 square meters, if the length of one side of the plot is *x*, the other side is $\frac{360}{x}$. This means the perimeter of the rectangle is

$$P(x) = 2(x + \frac{360}{x})$$

Here is a graph of *P(x)*:



The auxiliary lines show that if one side length x is between about 9 meters and 41 meters, the perimeter will be 100 meters or less.

To find these dimensions exactly, we can solve the following equation for x:

$$P(x) = 2\left(x + \frac{360}{x}\right) = 100$$

This is equivalent to the quadratic equation

$$x^2 - 50x + 360 = 0$$

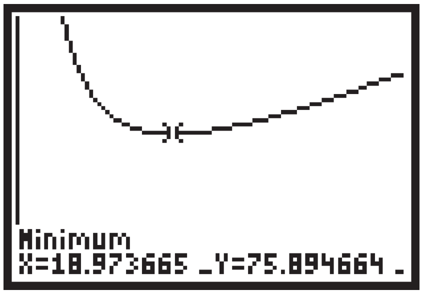
The solutions can be found by the quadratic formula

$$x = \frac{50}{2} \pm \frac{1}{2}\sqrt{(-50)^2 - 4(360)} = 25 \pm \frac{1}{2}\sqrt{1060} \approx 25 \pm 16.28$$

$$x = 8.72 \text{ or } x = 41.28$$

The solutions are approximately x = 8.72 meters and x = 41.28 meters. Since the area of the rectangle is 360 square meters, a side length of 8.75 meters of one side of the rectangle corresponds to a side length of $\frac{360}{8.72} \approx 41.28$ meters for the other side. This means that both solutions correspond to the same 8.72 by 41.28 rectangle. The perimeter will be less than or equal to 100 for width values between 8.72 meters and 41.28 meters.

2. To find which of the rectangular plots with an area of 360 square meters has the shortest perimeter, we have to find the value of x, for which $P(x)$ has a minimum value. Judging from the graph, this occurs at approximately x = 18.97 meters and $P(x)$ = 75.89 meters. One side of the rectangular plot is 18.97 meters.



Connection to TAKS:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

Since the area is 360 square meters, the length of the other side is approximately $\frac{360}{18.97} \approx 18.97$ meters. In other words, in our approximation from the graph we find that this rectangle of smallest perimeter is almost a square.

In fact, we can argue that it is exactly a square. Each horizontal line through the graph of *P(x)* intersects the graph in two places, where the two points on the *x*-axis correspond to the two sides of a rectangle with an area of 360 square meters. As the horizontal line is lowered, the two sides get closer together, so the corresponding rectangle is more nearly a square. At the lowest point where the horizontal line intersects the graph at all, the two points merge to one point, which corresponds exactly to a square.

The perimeter of that square is $4\sqrt{360} \approx 75.9$ meters. This is close to our earlier estimate from the graph.

Extension Questions:

- Suppose you want to enclose a rectangular area of exactly 500 square meters using exactly 96 meters of fence. Can you do it? If so, how? If not, why not?

  The most efficient way to use the fence would be first to make it a square. The length of a side would be $\frac{96}{4} = 24$ *meters, and the area would be* $24^2 = 576$ *square meters. Since this is more than 500 square meters, we can do it by making the rectangle not quite a square. Let one side of the rectangle be x meters. Then the other side is* $\frac{500}{x}$ *meters. Since the perimeter is 96 meters, we have*

  $$2\left(x + \frac{500}{x}\right) = 96$$

  This equation can be solved for x. It leads to the quadratic equation

  $$x^2 - 48x + 500 = 0$$

  The solutions are $x \approx 32.7$ *and* $x \approx 15.3$. *Since* $\frac{500}{32.7} \approx 15.3$, *these solutions both refer to the same rectangle. So the required rectangle has a length of 32.7 meters and a width of 15.3 meters.*

- What are the largest and smallest perimeters that can enclose a rectangular plot with a fixed area A_0?

  There is no largest perimeter, since by making the rectangle very long and thin, the perimeter can be made as large as desired while still enclosing the same area A_0.

But there is a smallest perimeter, namely the perimeter of a square with area A_0. This is $4\sqrt{A_0}$.

- Suppose you want to enclose a rectangular area of exactly 500 square meters, where the length of the rectangle is longer than the width by some fixed amount, *d* meters. Can you do this for any value of the amount *d* ? For only some values of *d* ? Or for no values of *d* ?

  Let the width be x meters. Then the length is x + d meters, and the area is x(x + d) square meters. Requiring this to be 500 square meters leads to the equation

  $$x(x + d) = 500$$

  This is equivalent to the quadratic equation

  $$x^2 + dx - 500 = 0$$

  Any solutions would be given by the quadratic formula:

  $$x = -\left(\frac{d}{2}\right) \pm \frac{1}{2}\sqrt{d^2 + 4(500)}$$

  Since x must be positive we must take the + sign of the ±. We get

  $$x = \sqrt{\left(\frac{d}{2}\right)^2 + 500} - \left(\frac{d}{2}\right)$$

  Since the quantity inside the square root is positive for any value of d, the square root is real for any d. Also, the value for x is always positive, since

  $$\sqrt{\left(\frac{d}{2}\right)^2 + 500} > \sqrt{\left(\frac{d}{2}\right)^2} = \frac{d}{2}$$

  So the answer is that we <u>can</u> always make such a rectangle for any value of d.

- Solve the initial problem generally. If you wish to enclose a rectangular plot of area exactly A_0 square meters, and can use up to P_0 meters of fencing to do the job, what are possible dimensions of the plot?

  Since the desired area of the rectangle is A_0 square meters, if the length of one side of the plot is x, the other side is $\frac{A_0}{x}$. This means the perimeter of the rectangle is

  $$P(x) = 2(x + \frac{A_0}{x})$$

The requirement that the perimeter be less than P_0 meters leads to the inequality

$$2\left(x + \frac{A_0}{x}\right) \le P_0$$

We can write this inequality as

$$2x^2 - P_0x + 2A_0 \le 0$$

The solutions can be found by first solving the corresponding quadratic equation $2x^2 - P_0x + 2A_0 = 0$ *using the quadratic formula. Its solutions are*

$$x = \frac{P_0}{4} - \frac{1}{4}\sqrt{P_0^2 - 16A_0} \text{ and } x = \frac{P_0}{4} + \frac{1}{4}\sqrt{P_0^2 - 16A_0}$$

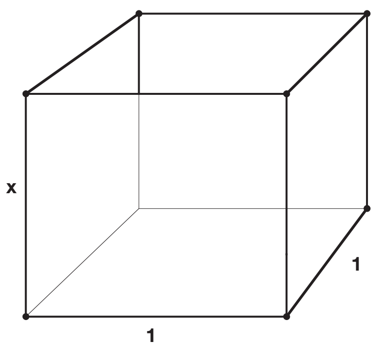
The graph of the quadratic function $f(x) = 2x^2 - P_0x + 2A_0$ *is a parabola that opens up. The graph will be below the x-axis between the two roots. Therefore, the rectangle will have a perimeter less than P_0:*

$$\frac{P_0}{4} - \frac{1}{4}\sqrt{P_0^2 - 16A_0} < x < \frac{P_0}{4} + \frac{1}{4}\sqrt{P_0^2 - 16A_0}$$

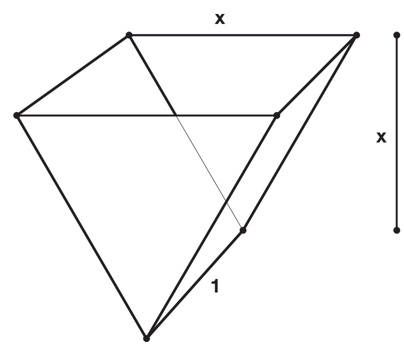
Comparing Volumes

1. Joe and Clara are designing containers for a school art project. The containers are described below. They have a common base length of one unit, but other dimensions—indicated by the variable *x*—have not been determined.

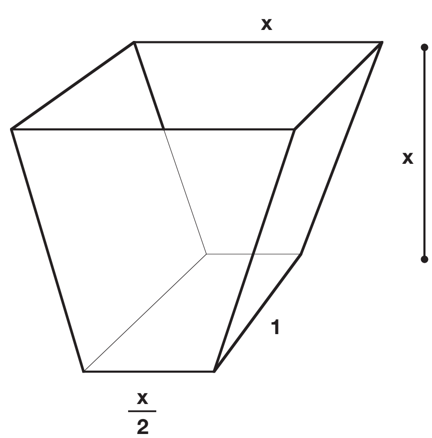
- Rectangular prism with a square base
- Side length 1 foot
- Depth *x*



- Isosceles triangular prism
- Equal top width and depth, *x*
- Base length 1 foot.



- Isosceles trapezoidal prism
- The shorter base is $\frac{x}{2}$
- The longer base and trapezoid height are both *x* units in length
- Base length 1 foot

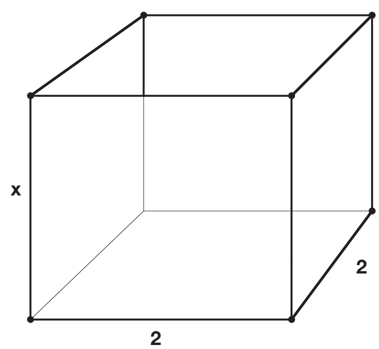


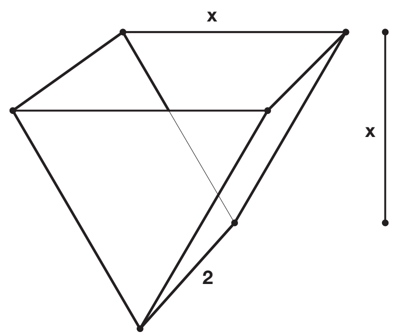
Determine a formula for the volume of each container in terms of *x,* measured in feet. Compare the volumes of the three containers for varying values of *x.*

2. Find a formula for the volumes of each container if the base length 1 foot is changed to 2 feet.

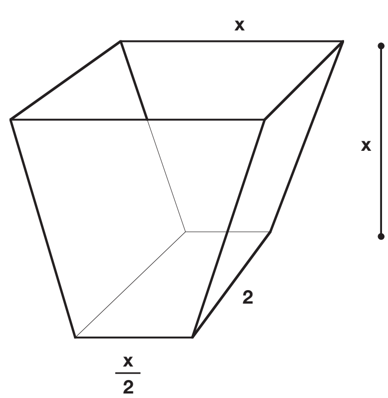
- Rectangular prism with a square base
- Side length 2 feet
- Depth *x*

- Isosceles triangular prism
- Equal top width and depth, *x*
- Base length 2 feet





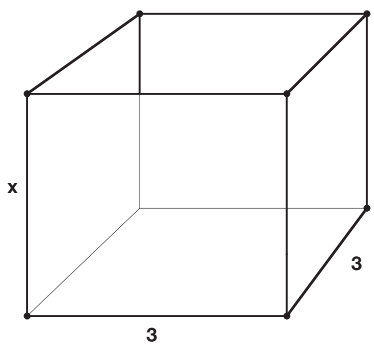
- Isosceles trapezoidal prism
- The shorter base is $\frac{x}{2}$
- The longer base and trapezoid height are both *x* units in length
- Base length 2 feet



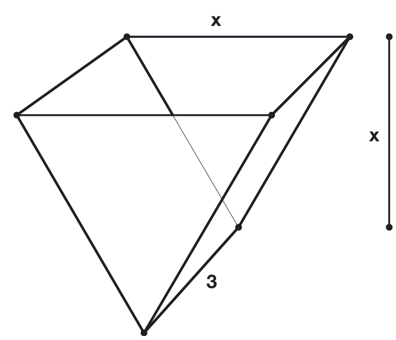
Compare the volumes of the three different containers.

3. Find a formula for the volume of each container if the base length 1 foot is changed to 3 feet.

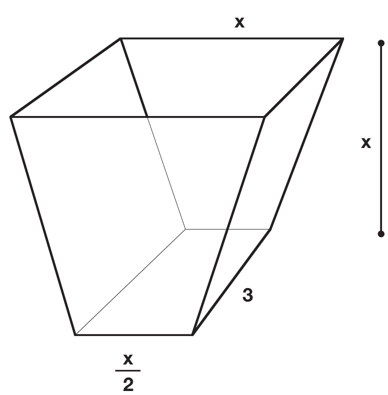
- Rectangular prism with a square base
- Side length 3 feet
- Depth x



- Isosceles triangular prism
- Equal top width and depth, x
- Base length 3 feet

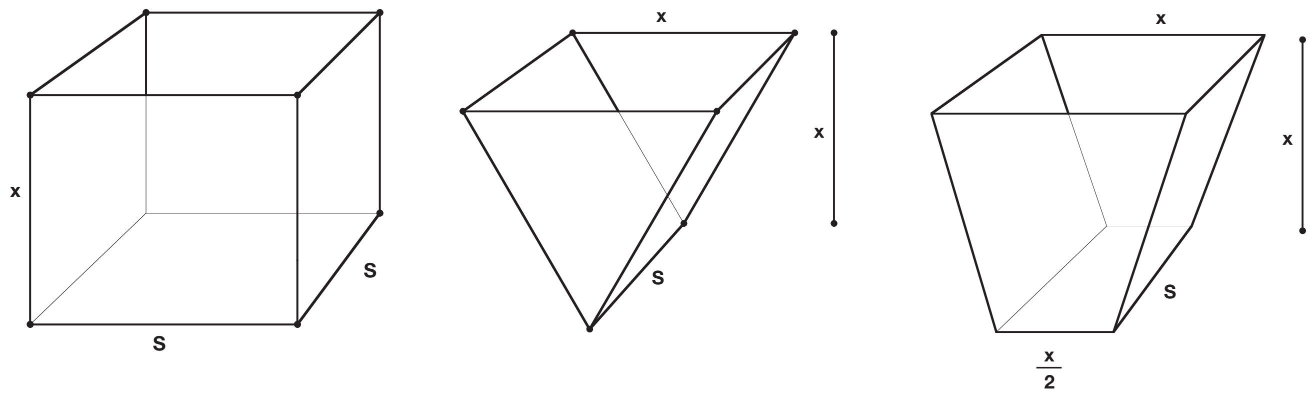


- Isosceles trapezoidal prism
- The shorter base is $\frac{x}{2}$
- The longer base and trapezoid height are both x units in length
- Base length 3 feet



Compare the volumes of the three containers.

4. Consider the general situation where the containers each have a base length of *s* feet and the indicated measurements. Write formulas for the volumes of the three solids.



Use systems of equations and multiple representations to determine the relationship between *x* and *s* when the volumes of any two of the containers are equal.

Notes

Materials:

Graphing calculator

Algebra II TEKS Focus:
(2A.3) Foundations for functions. The student formulates systems of equations and inequalities from problem situations, uses a variety of methods to solve them, and analyzes the solutions in terms of the situations.

The student is expected to:

(A) analyze situations and formulate systems of equations in two or more unknowns or inequalities in two unknowns to solve problems.

(B) use algebraic methods, graphs, tables, or matrices, to solve systems of equations or inequalities.

Additional Algebra II TEKS:

(2A.3) Foundations for functions. The student formulates systems of equations and inequalities from problem situations, uses a variety of methods to solve them, and analyzes the solutions in terms of the situations.

The student is expected to:

(C) interpret and determine the reasonableness of solutions to systems of equations or inequalities for given contexts.

Scaffolding Questions:

- How do you find the volume of a prism?
- What cross-sections of the prisms can you use to help you find their volumes?
- What geometric properties can you use?
- Once you have the volume formulas for the containers, what representations can you use to compare the three volumes?
- When you use algebra to solve a system with fractional or decimal coefficients, what can you do to make the system simpler to solve?

Sample Solutions:

1. Let V_1 = the volume of the square prism.

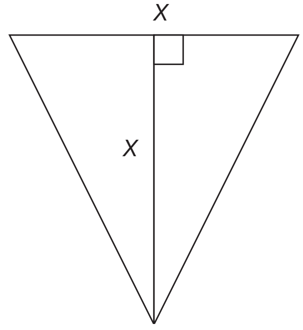
   V_2 = the volume of the triangular prism.

   V_3 = the volume of the trapezoidal prism.

   For all three containers, we can think of volume as base area times prism height.

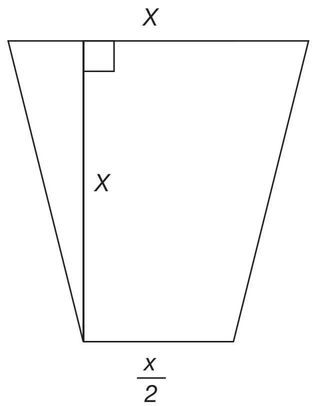
   A. The volume of the square prism is $V_1 = x(1)^2 = 1x$.

   B. To get the base area of the triangular prism, we need to look at a cross-section of the prism, showing the isosceles triangle.



The area of the triangle is $A = \frac{1}{2}x^2$. The volume of the triangular prism is $V_2 = A(1) = \left(\frac{1}{2}x^2\right)1 = 0.5x^2$.

C. For the base of the trapezoidal prism, look at a cross section of the prism showing the trapezoid.



The area of the trapezoid is

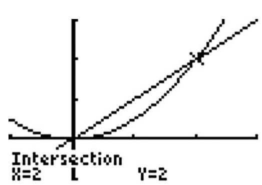
$$A = \frac{x}{2}\left(x + \frac{x}{2}\right) = \frac{x}{2}\left(\frac{3x}{2}\right) = 0.75x^2$$

The volume of the trapezoidal prism is

$$V_3 = \left(0.75x^2\right)1 = 0.75x^2$$

To compare the volumes of the containers to each other, we can graph the volumes.

Here are the graphs of the volumes of the square and triangular prisms:



The square prism has the greater volume until the depth is 2 units. At 2 units they have equal volume, 2 cubic units. For depths greater than 2 units the triangular prism has the greater volume.

Connection to TAKS:

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

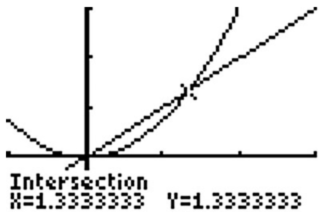
Objective 5: The student will demonstrate an understanding of quadratic and other nonlinear functions.

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Here are the graphs of the volumes of the square and trapezoidal prisms:



The square prism has the greater volume until the depth is $\frac{4}{3}=1\frac{1}{3}$ units. For a depth of $\frac{4}{3}=1\frac{1}{3}$ units they are equal in volume. For depths greater than $\frac{4}{3}=1\frac{1}{3}$ units, the trapezoidal prism has the greater volume.

By comparing the graphs or the formulas for the triangular and trapezoidal prisms, we see that the trapezoidal prism will always have the greater volume.

2. The volume of the square prism is $V_1 = 2^2 x = 4x$.

The triangular prism has the same cross-section shown in problem 1, the isosceles triangle.

The area of the triangle is $A = \frac{1}{2}x^2$. The volume of the triangular prism is

$$V_2 = A(2) = \frac{1}{2}x^2 2 = x^2$$

The trapezoidal prism has the same cross-section shown in problem 1, the trapezoid.

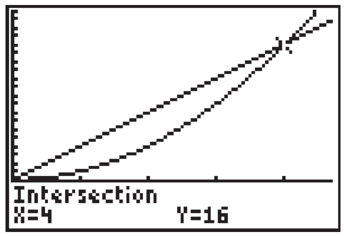
The area of the trapezoid is

$$A = \frac{x}{2}\left(x + \frac{x}{2}\right) = \frac{x}{2}\left(\frac{3x}{2}\right) = 0.75x^2$$

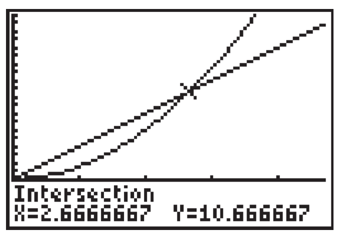
The volume of the trapezoidal prism is $V_3 = \left(0.75x^2\right)2 = 0.75(2)x^2 = 1.5x^2$.

To compare the volumes, compare the graphs.

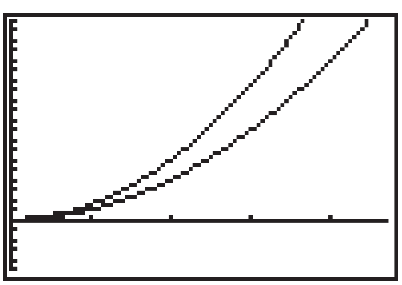
The volume function for the rectangular prism graph (the line) intersects the graph of volume function for the triangular prism at (4, 16). The volume of the triangular prism is greater for values of *x* greater than or equal to 4.



The volume function for the rectangular prism graph (the line) intersects the graph of volume function for the trapezoidal prism at $\left(2\frac{2}{3}, 10\frac{2}{3}\right)$. The volume of the trapezoidal prism is greater for values of *x* greater than or equal to $2\frac{2}{3}$.



The graphs of the volume functions for the triangular prism and the trapezoidal prism appear to intersect at *x* = 0; however, the table shows that the trapezoidal prism volume is always greater.



X	Y2	Y3
0	0	0
.1	.01	.015
.2	.04	.06
.3	.09	.135
.4	.16	.24
.5	.25	.375
.6	.36	.54

X=0

This comparison is also evident by comparing the formulas for these volumes. The volume of the pyramid is $V_2 = 1.5x^2$. The volume of the trapezoidal solid is $V_3 = 2.25x^2$.

$$1.5x^2 < 2.25x^2$$
$$V_2 < V_3$$

The volume of the pyramid is always less than the volume of the trapezoidal prism.

3. The volume of the square prism is $V_1 = x3^2 = 9x$.

The triangular prism has the same cross-section shown in problem 1, the isosceles triangle.

The area of the triangle is $A = \frac{1}{2}x^2$. The volume of the triangular prism is

$$V_2 = A(3) = \frac{1}{2}x^2 3 = 1.5x^2$$

The trapezoidal prism has the same cross-section shown in problem 1, the trapezoid.

The area of the trapezoid is

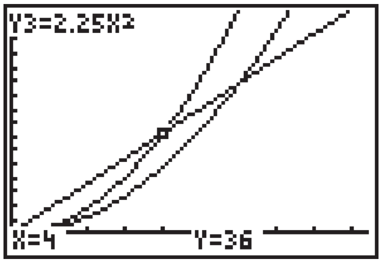
$$A = \frac{x}{2}\left(x + \frac{x}{2}\right) = \frac{x}{2}\left(\frac{3x}{2}\right) = 0.75x^2$$

The volume of the trapezoidal prism is

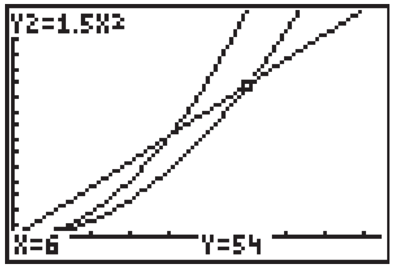
$$V_3 = \left(0.75x^2\right)3 = 0.75(3)x^2 = 2.25x^2$$

To compare the volumes, compare the graphs.

The volume function for the rectangular prism graph (the line) intersects the graph of volume function Y_3 for the trapezoidal prism at (4, 36). The volume of the trapezoidal prism is greater for values of *x* greater than or equal to 4.



The volume function for the rectangular prism graph (the line) intersects the graph of volume function Y_2 for the triangular prism at (6, 54). The volume of the trapezoidal prism is greater for values of x greater than or equal to 6.



The graphs of the volume functions for the triangular prism and the trapezoidal prism appear to intersect at $x = 0$; however, the table shows that the trapezoidal prism volume is always greater.

X	Y2	Y3
0	0	0
.1	.015	.0225
.2	.06	.09
.3	.135	.2025
.4	.24	.36
.5	.375	.5625
.6	.54	.81

X=0

4. Consider the general case. The volume of the square prism is $V_1 = xs^2 = s^2x$.

The triangular prism has the same cross-section shown in problem 1, the isosceles triangle.

The area of the triangle is $A = \frac{1}{2}x^2$. The volume of the triangular prism is

$$V_2 = As = \frac{1}{2}x^2 s = 0.5sx^2$$

The trapezoidal prism has the same cross-section shown in problem 1, the trapezoid.

The area of the trapezoid is

$$A = \frac{x}{2}\left(x + \frac{x}{2}\right) = \frac{x}{2}\left(\frac{3x}{2}\right) = 0.75x^2$$

The volume of the trapezoidal prism is $V_3 = \left(0.75x^2\right)s = 0.75sx^2$.

For the square prism, the volume V_1 depends linearly on the depth, x. For the triangular and trapezoidal prisms, the volumes are quadratic functions of the depth.

To compare the volumes of the square and triangular prisms, we can solve the system:

$$\text{Let } V_1 = s^2x \text{ and } V_2 = 0.5sx^2$$
$$\text{Then } 0.5sx^2 = s^2x$$
$$0.5sx^2 - s^2x = 0$$
$$sx^2 - 2s^2x = 0$$
$$sx(x - 2s) = 0$$
$$sx \neq 0 \text{ so } x - 2s = 0$$
$$x = 2s$$

The volumes of the square and triangular prisms are equal when the value of *x* is twice the value of the length of the base, *s*.

To compare the volumes of the square and trapezoidal prisms,

$$\text{Let } V_1 = s^2x \text{ and } V_3 = 0.75sx^2$$
$$\text{Then } 0.75sx^2 = s^2x$$
$$0.75sx^2 - s^2x = 0$$
$$sx(0.75x - s) = 0$$
$$sx \neq 0 \text{ so x} = \frac{s}{0.75} = \frac{4}{3}s$$

The volume of the square prism and the volume of the trapezoidal prism are equal when the value of *x* is four-thirds the value of s.

The system for the triangular and trapezoidal prisms is

$$0.5sx^2 = 0.75sx^2$$
$$0.25sx^2 = 0$$

Since $s \neq 0$ and $x \neq 0$, this system has no solution. This means that the volume of the square prism and the volume of the trapezoidal prism will never be equal for the same values of *x* and *s*. In the three previous examples it was shown that the volume of the trapezoidal prism is always greater than the volume of the triangular prism.

Extension Questions:

- Describe some mathematical concepts that helped you solve these problems.

  We had to recall how to find volumes of solids, such as cubes and prisms. We had to be able to visualize cross-sections of solids so that we could get the correct volume formulas for the three solids.

  Once we had formulas for the volumes of the three solids, we had to think about systems of equations and how to solve them. We used tables, graphs, and algebraic methods to solve the systems.

- Describe how you solved the systems.

  We graphed the systems and determined the points of intersection. We solved the systems algebraically using the Substitution Method.

- Describe a practical situation in which you would be concerned about the volumes of these 3 containers.

  Sample solution:

  Suppose you are designing containers for packaging, and the length of the containers is s units.

  Compare the dimensions of the square and triangular prisms if we want them to have equal volume. The dimensions of the square prism would be s units long by s units wide by 2s units deep. The triangular prism would be s units long by 2s units wide by 2s units deep.

  Compare the dimensions of the square and trapezoidal prisms if we want them to have equal volume. The dimensions of the square prism would be s units long by s units wide by $\frac{4}{3}s$ *units deep. The trapezoidal prism would be s units long. The shorter base of the trapezoid would be* $\frac{2}{3}s$ *units wide. The longer base and the depth would be* $\frac{4}{3}s$ *units.*

  If we needed to ship a number of these containers in, for example, a truck, we would want to use the container shape that would let us pack the greatest number of containers in the truck.